NC STATE Math 241-005 - Study Guide - Spring 2021
Chapter 3

3.1: Intro to Diff. Eq.

- Be able to verify if a given function is a solution to a Diff.Eq.
- Identify the order of a Diff.Eq.
- Know the definition of a general solution to a Diff.Eq.
- Be able to distinguish between: First Order Separable Diff.Eq., Second Order Linear Homogeneous Diff.Eq., and Second Order Linear Non-Homogeneous Diff.Eq..

3.1: Slope Fields

- Know how to sketch a small Slope Field (Note: You will NOT be asked to sketch one on the test)
- Know the definition of an equilibrium solution and be able to identify it on a slope field
- Match a Diff.Eq. to it's Slope Field

3.1: Euler's Method

- Given an initial condition and step-size, approximate points on a solution curve

3.2: Separable Diff. Eq.

- Identify when a Diff.Eq. is separable
- Use the Separation of Variables technique to find the general solution
- Use the Separation of Variables technique to solve an Initial Value Problem (IVP)

3.2 & 3.3 Applications of 1st Order Separable Diff. Eq.

- Given a family of curves, use the Orthogonal Trajectory technique to find an orthogonal family of curves.
- Given an observation about the rate of change of a population/substance/temperature, be able to setup an Initial Value Problem (IVP) that models the scenario.
- Given the description of the model, use the solution to find a population/substance/temperature at a given time.
- Use Separation of Variables to find the amount of substance in a tank at a given time.

3.4 2nd Order Linear Homogeneous

- Know the definition of the auxiliary/charachteristic equation
- Be able to find the general solution
- Be able to solve an initial value problem

3.5 2nd Order Linear Non-Homogeneous

- Be able to find the particular solution
- Be able to find the general solution
- Be able to solve an initial value problem

3.6 Applications of 2nd Order Linear Diff.Eq.

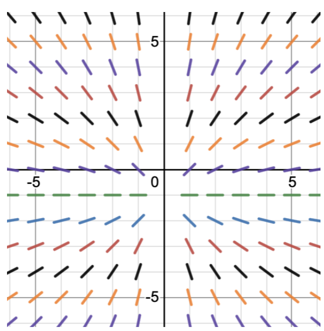
- Be able to find the position of a mass at the end of a spring at time t

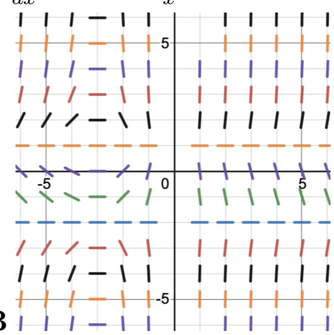
Note: Circuits will NOT be tested on this Exam.

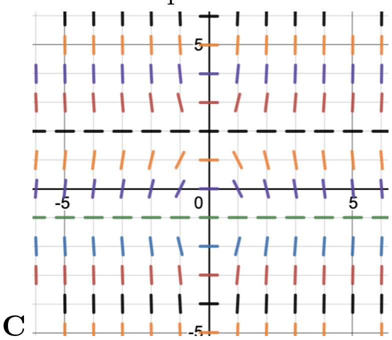
Exercises: (Note: the level of difficulty of the *Challenge* questions will not be on your test)

1. True or False: $y = -\frac{1}{2}x\cos(x)$ is a solution to $\frac{d^2y}{dx^2} + y = \sin(x)$.

2. Find the equilibrium solutions of $\frac{dy}{dx} = \frac{(y+2)(y-1)(x+3)}{x}$ and match it to its slope field.

A 

B 

C 

3. Use Euler's method with a step size of 0.5 to compute $y(2.5)$, where $y(x)$ is a solution to the IVP:

$$y' = 3x - y, \qquad y(1) = 0$$

4. Use Euler's method with a step size of 0.5 to compute $y(2)$, where $y(x)$ is a solution to the IVP:

$$\frac{dy}{dx} = \sqrt{yx}, \qquad y(1) = 2$$

5. Find the general solution to the following Differential Equation

$$\frac{1}{(x^2+1)}\frac{dy}{dx} = \frac{1}{xy}$$

6. Find the solution to the differential equation: $\frac{dy}{dx} = 4x^3y$, with $y(0) = 5$.

7. Find the orthogonal trajectory to $y = ke^{-x}$, where k is a constant.

8. A \$200,000 Ferrari was purchased in 2010. In 2015 it had a value of \$150,000. Assuming the value is decreasing exponentially, find the value of the Ferrari in 2021.

9. A hot cup of soup cools from $140°F$ to $110°F$ in 20 mins. when placed in a room of $72°F$.

   (a) What is T after 20 more minutes?

   (b) When will the soup be $80°F$?

10. 'Cat island' is Japan has a maximum supportable population of 1,000,000 cats. In 1990 there were 731,327 cats. In 2000 there were 781,926.

    (a) What is the predicted population in 2020?

    (b) What happens in the long-run?

11. A car's cooling system has a capacity of 15 gallons. Initially the system contains a mixture 22 of 5 gallons of antifreeze and 10 gallons of water. Antifreeze runs into the system at a rate of 2 gallons per minute. The well-mixed solution leaves at the same rate. How much antifreeze is in the system at the end of 30 minutes? (There is no need to simplify your final answer past simple algebra.)

12. (Challenge) A tank initially contains 100L of water in which 3 kg. of salt has been dissolved. Pure water enters the tank at a rate of $4L/min$. The solution is well-mixed and drains from the tank at a rate of $2L/min$. Find the amount of salt in the tank at time t.

13. (Challenge) Given the differential equation for the logistic model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

Use separation of variables and partial fractions to show $P(t) = \frac{kP_0}{P_0 + (k - P_0)e^{-rt}}$.

14. State the 3 cases of general solutions to the Diff. Eq.

$$ay'' + by' + cy = 0$$

if a, b, c are all constants.

15. Find the general solution to $y'' - 10y' + 26y = 0$.

16. Given the differential equation below,

$$y'' - 8y' + 16y = 0$$

   (a) State the auxiliary equation.
   (b) State the general solution.
   (c) Find the solution given the following initial conditions: $y(0) = 5$ and $y'(0) = 3$.

17. Solve $y'' + 9y = 0$, with $y(0) = 3, y(\frac{\pi}{6}) = 7$

18. For each of the following differential equations, state the form of the particular solution. (Do NOT solve for any of the variables).

   (a) $y'' - 3y' + 2y = e^{2x}$
   (b) $y'' - 3y' + 2y = \sin(3x)$
   (c) $y'' + 2y' + y = 3e^{-x} + \sin(2x) + 5$

19. Find the general solution to $y'' + 3y' - 10y = 5e^{2x}$

20. Solve $y'' - y' = xe^x$, with $y(0) = 0$, $y'(0) = 0$

21. A spring with mass of 1 kg. has a damping constant $6kg/s$. A force of $2N$ is required to keep the spring stretched $0.2m$. beyond its natural length. The spring is compressed $1m$. from its natural length and released.

   (a) Find the position of the mass at time t.
   (b) What kind of damping is this?

Solutions:

1. True
2. Equilibrium solutions: $y = 1, y = -2$.
   Slope-field: B.
3. $y(2.5) = 4.5$
4. $y(2) = 3.715$
5. $y^2 = x^2 + 2\ln|x| + C$
6. $y = 5e^{x^4}$
7. $y^2 = 2x + C$
8. $y(11) = (200,000)e^{\frac{11}{5}\ln(\frac{3}{4})} \approx \$106,209.80$
9. (a) $93.2°F$
   (b) 73.55 mins
10. (a) $P(30) = \frac{1{,}000{,}000}{1+\left(\frac{1{,}000{,}000}{731{,}327}-1\right)e^{-0.02756(30)}} \approx 861,530$ cats
    (b) $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{1{,}000{,}000}{1+\left(\frac{1{,}000{,}000}{731{,}327}-1\right)e^{-0.02756t}} = \frac{1{,}000{,}000}{1+\left(\frac{1{,}000{,}000}{731{,}327}-1\right)0} = 1,000,000$ cats (the carrying capacity)
11. $y(30) = -10e^{-\frac{2}{15}(30)} + 15 = -10e^{-4} + 15$ gallons
12. This one is challenging because the volume of the tank is changing every minute. (Since 4 L are coming in each minute, but only 2 L are leaving). So the water level is rising, and each minute the volume is increasing by 2 L. So we model the volume of the tank as $100 + 2t$. That means the initial value problem for this is:

$$\frac{dy}{dt} = 4(0) - 2\left(\frac{y}{100+2t}\right) \qquad y(0) = 3$$

    Solving this, you'll get $y(t) = \frac{150}{50+t}$
13. (answer omitted, Challenge question)
    Hint: $rP\left(1 - \frac{P}{k}\right) = rP\left(\frac{k-P}{k}\right)$. Then do separation of variables.

14. (a) If $b^2 - 4ac > 0$, then we get 2 distinct real roots r_1 and r_2, so: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
    (b) If $b^2 - 4ac = 0$, then we get 1 repeated real root r, so: $y = C_1 e^{rx} + C_2 x e^{rx}$
    (c) If $b^2 - 4ac < 0$, then we get complex roots $\alpha \pm \beta i$, so: $y = e^{\alpha x}[C_1 \cos(\beta x) + C_2 \sin(\beta x)]$
15. $y = e^{5x}[C_1 \cos(x) + C_2 \sin(x)]$
16. (a) $r^2 - 8r + 16 = 0$
    (b) $y = C_1 e^{4x} + C_2 x e^{4x}$
    (c) $y = 5e^{4x} + (-17)xe^{4x}$
17. $y = 3\cos(3x) + 7\sin(3x)$
18. (a) $y_p = Axe^{2x}$
    (b) $y_p = A\sin(3x) + B\cos(3x)$
    (c) $y_p = Ax^2 e^{-x} + B\cos(2x) + C\sin(2x) + D$
19. $y = C_1 e^{-5x} + C_2 e^{2x} + \frac{5}{7} x e^{2x}$
20. $y = -1 + e^x + \frac{1}{2} x^2 e^x - xe^x$
21. (a) $x(t) = e^{-3t}(\cos(t) + 3\sin(t))$
    (b) under-damping