Math 241-003 - Study Guide - Fall 2021

Chapter 1

0.1: Prerequisites

- Take derivatives (chain rule, product rule, quotient rule, power rule, etc.)
- Integrate basic functions (polynomials, exponential, logarithm, trig., etc.)
- Use u-substitution
- Use integration by parts
- Compute $\sin(x)$, $\cos(x)$, and $\tan(x)$ for $x = 0, \pi/2, \pi$
- Recall $\sin^2(u) + \cos^2(u) = 1$
- Recall the area of circles, rectangles, and squares
- Recall similar triangles, $y = mx + b$ form, and the xy-equation for a circle centered at (a, b).

1.1: Arclength

- Setup the 3 Forms of arclength
  (What are they? When should you use each one?)
- Evaluate arclength integrals
  Techniques: perfect squares, trig. identities, and u-sub
- Summarize the process of creating the arclength formulas

1.2: Average Values

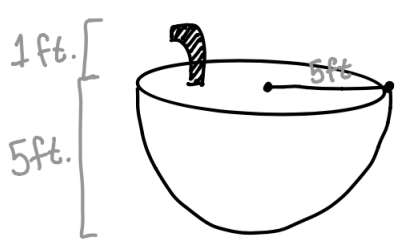
- Find the average value of a continuous function over a given interval
- Summarize the process of creating the average value formula
- Consider how to find the average value of a positive discontinuous function over a given interval
- Represent the average value of a continuous function geometrically
- State the Mean Value Theorem
  (What does it guarantee? Could you summarize it in your own words?)
- Find a c in a given interval so that $f(c) = f_{avg}$

1.3: Physics Applications

- Find the amount of work done pulling, stretching, compressing a rope or spring along a straight line
- Use Hooke's Law to compute a spring constant.
- Find the amount of work done pumping liquid out of a tank, or above a tank
- Find the amount of pressure due to liquid on a flat object in a tank
- Find the center of mass of a lamina
- Graph the center of mass of a lamina

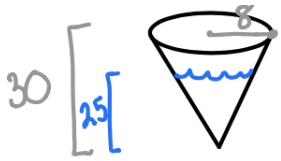
EXERCISES:

1. A circle of radius r has the parametric equation $(r\cos(t), r\sin(t))$ for $0 \leq t \leq 2\pi$.
   (a) Find the arclength of the circle with radius r
   (b) Compare your equation in part (a) to the circumference of a circle. What do you notice?
2. Let $f(x) = \frac{x^3}{6} + \frac{1}{2x}$. Find the arclength of the curve for $1 \leq x \leq 4$.
3. (a) State all three arclength formulas
   (b) Briefly outline the steps you would use to create one of the equations you listed in part (a).
4. Let $f(x) = 8x - x^2$ on $[0, 4]$
   (a) Find f_{avg}.
   (b) State the Mean Value Theorem for Integration.
   (c) Find a specific c that verifies the Mean Value Theorem for Integration.
5. Let $f(x) = \sin(x) + 1$ on $[0, 2\pi]$
   (a) Find f_{avg}.
   (b) Sketch a picture of $f(x)$. Does your answer in part (a) make sense geometrically? Why?
6. Let $f(x) = e^{2x} + 1$ on $[1, 3]$
   (a) Find f_{avg}
   (b) Find a c in $[1, 3]$ so that $f(c) = f_{avg}$.
7. A rope that weighs $\frac{1}{2}lb/ft$ is used to lift a bucket of water weighing $5lbs.$ from a $12ft.$ well. Find the work done lifting the bucket to the top of the well.
8. A spring whose equlibrium length is $4m.$ extends to a length of $8m.$ when a force of $2N$ is applied.
   True or False: Since $F(x) = kx$ then $2N = k(8m.)$. Then the spring constant is $k = \frac{1}{4}N/m.$.
9. $2J$ of work is required to stretch a spring from its natural length of $30cm$ to $42cm$. What is the amount of work needed to stretch the spring from $35cm$ to $40cm$?
10. A spring has a natural length of $8in.$ A force of $6lbs.$ is required to hold the spring at $12in.$ Find the work done stretching the spring from $10in.$ to $16in.$.
11. The tank shown is full of water. Given that water weighs 62.4 lb/ft^3, SETUP an equation to find the work required to pump the water out of a spout located 1 $ft.$ above the tank.

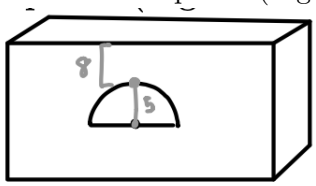


← lower half of a sphere

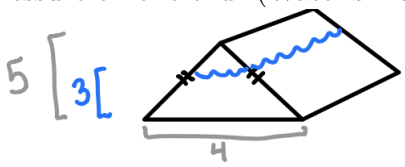
12. An inverted conical tank (vertex at the bottom) has a radius of $8m.$ and height of $30m.$ It is filled with $25m.$ of water, which has a mass density of $1000kg/m^3$. SETUP an equation to find the amount of work used to pump all the water to the top of the tank.



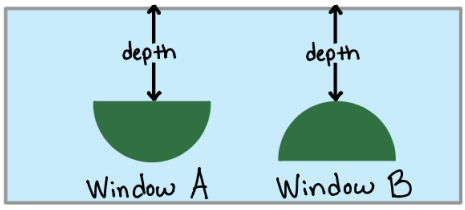
13. A half-circular plate is submerged 8 ft. into a tank filled with water. If the radius of the plate is 5 ft. and water has a weight density of 62.4 lb/ft^3, SETUP an equation to find the force on one end of the plate. (Figure is shown below)



14. A trough has a vertical cross section of an isosceles triangle that is $4m.$ across and $5m.$ tall. If the trough is filled with to $3m.$ with water, SETUP an equation to the force due to water pressure on one end. (Water's mass density is $1000kg/m^3$.)



15. Two identical semicircular windows are placed at the same depth in the vertical wall of an aquarium (see figure).
True or False: Window B will have a greater force due to hydrostatic pressure.



16. Consider the homogeneous lamina bounded by $y = 2\sin(2x)$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.
    (a) Find the center of mass
    (b) Plot the region and your center of mass on the xy-plane

17. Consider the homogeneous lamina bounded by $y = x^2 + 1$, $y = 0$, $x = -2$, and $x = 2$.
    (a) Find the center of mass
    (b) Plot the region and your center of mass on the xy-plane
    (c) Do you notice anything strange in (b)?

Solutions:

1. (a) $\mathscr{L} = 2\pi r$
   (b) The circumference of a circle is $2\pi r$. This matches our answer from (a). So (the circumference of a circle)=(the arclength of a circle)

2. $\left(\frac{4^3}{6} - \frac{1}{8}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$

3. (a) $\mathscr{L} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

   $\mathscr{L} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

   $\mathscr{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

   (b) Approximate the curve using straight line segments
   $\downarrow$
   Add up the lengths of the segments in the approximation
   $\downarrow$
   Take infinitley many segments to get the exact length of the curve

4. (a) $f_{avg} = \frac{32}{3}$
   (b) If f is continuous on $[a, b]$, then there exists at least one c in $[a, b]$ so that $f(c) = f_{avg}$.
   (c) $c = 1.69$

5. (a) $f_{avg} = 0$
   (b) Yes, the function spends an equal amount of time above and below the line $y = 1$, so we should expect the average value to be 1.

6. (a) $f_{avg} = \frac{e^6}{4} + 1 - \frac{e^2}{4}$
   (b) $c = \frac{1}{2} \ln\left(\frac{e^6}{4} - \frac{e^2}{4}\right)$

7. $W = 96\ ft \cdot lbs.$

8. False.
   Recall that x =(distance beyond natural length). So we should have: $F(x) = kx$ makes $2N = k(4m.)$ Then the spring constant is $k = \frac{1}{2}N/m$.

9. $W = \frac{4}{(.12)^2}\left(\frac{(0.1)^2}{2} - \frac{(0.05)^2}{2}\right) = 1.0417$ (units: J)

10. $W = 9\left(\frac{2}{3}\right)^2 - 9\left(\frac{1}{6}\right)^2 = \frac{15}{4} = 3.75$ (units: $ft \cdot lbs.$)

11. - If you set the origin at the top-center of the tank
    $W = \int_{-5}^{0} (62.4)\pi(25 - y^2)(-y + 1)\, dy$
    - If you set the origin at the bottom-center of the tank
    $W = \int_{0}^{5} (62.4)\pi(25 - (y - 5)^2)(6 - y)\, dy$

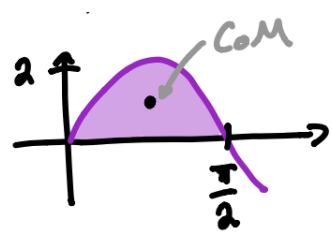
12.
- If you set the origin at the top-center of the tank
$$W = \int_{-30}^{-5} (1000 \cdot 9.8)\pi \left(\frac{8}{30}(30+y)\right)^2 (-y)\ dy$$
- If you set the origin at the bottom-center of the tank
$$W = \int_{0}^{25} (1000 \cdot 9.8)\pi \left(\frac{8}{30}y\right)^2 (30-y)\ dy$$

13.
- If you set the origin at the top-center of the tank
$$HF = \int_{-13}^{-8} (62.4)(2\sqrt{25-(y+13)^2})(-y)\ dy$$
- If you set the origin at the center of the circle
$$HF = \int_{0}^{5} (62.4)(2\sqrt{25-y^2})(13-y)\ dy$$

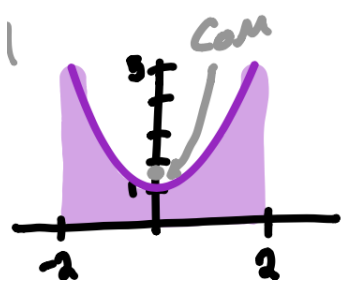
14.
- If you set the origin at the top-center of the tank
$$HF = \int_{-5}^{-2} (1000 \cdot 9.8)\left(-\frac{4}{5}y\right)(-y)\ dy$$
- If you set the origin at the bottom-center of the tank
$$HF = \int_{0}^{3} (1000 \cdot 9.8)\left(\frac{4}{5}(5-y)\right)(3-y)\ dy$$

15. True. Since pressure varies by depth, that means having more area at the bottom will produce more pressure on the plate. Since Window B will have more pressure being exerted on it, it has a greater force due to hydrostatic pressure.

16. (a) CoM $= (\bar{x}, \bar{y}) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

(b) 

17. (a) CoM $= (\bar{x}, \bar{y}) = \left(0, \frac{103}{70}\right) \approx (0, 1.47)$

(b) 

(c) You should notice that the center of mass is not inside the lamina. That's ok! That just means there is no spot where the lamina balances.