Determination of a strictly convex and non-trapping Riemannian Manifold from partial travel time data

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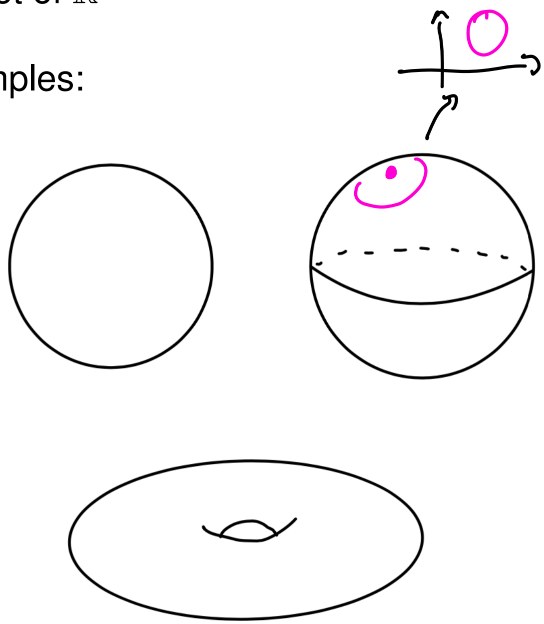
Manifolds

An n-dimensional **smooth manifold** is a Hausdorff and second countable topological space, with a maximal atlas.

smooth structure

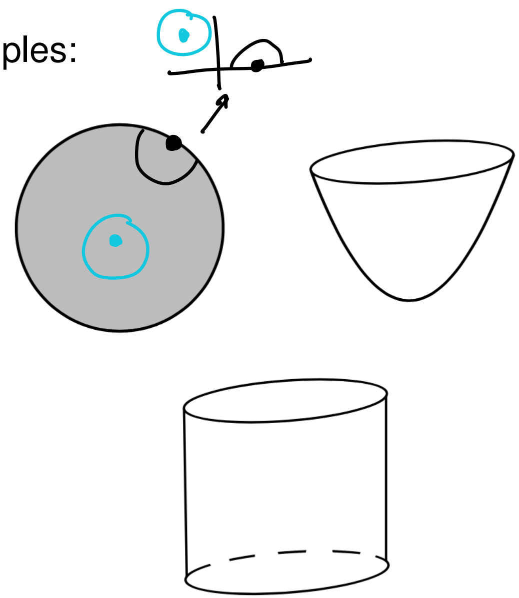
A smooth manifold without boundary:
Each neighborhood is diffeomorphic to a subset of $\mathbb{R}^n$

Examples:



A smooth manifold with boundary:
Each neighborhood is diffeomorphic to a subset of the upper-half of $\mathbb{R}^n$.

Examples:

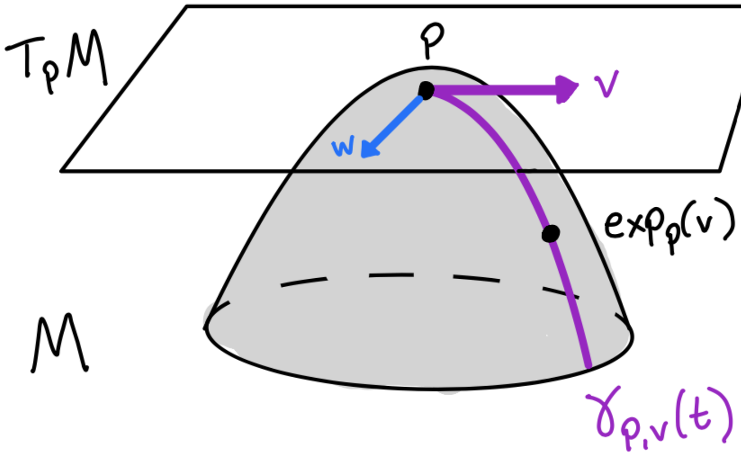


Riemannian Structure

An n-dimensional **Riemannian manifold** (M, g) is a smooth manifold M equipped with Riemannian metric g.

From g, we get :

1. $\langle v, w \rangle_g$ for $v, w \in T_pM$.
2. $\|v\|_g = \sqrt{\langle v, v \rangle_g}$.
3. Gradients, *Grad f*.
4. Geodesics, $\gamma_{p,v}(t)$.
5. Exponential map, $exp_p(v)$.
6. Distances, $d(p, q)$.

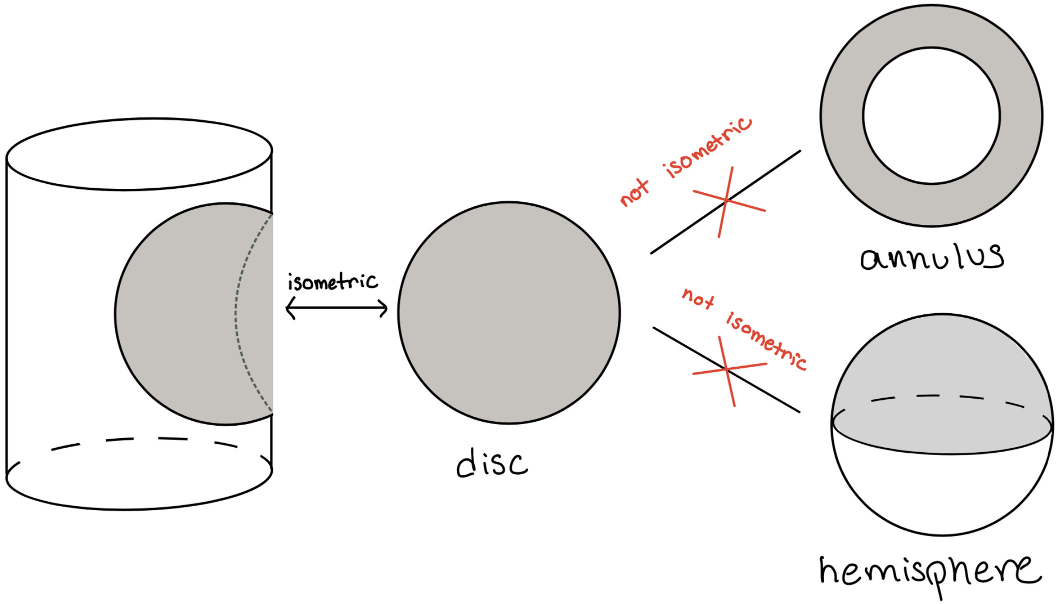


Isometry

(M_1, g_1) and (M_2, g_2) be Riemannian manifolds with boundary. An **isometry** ψ is a diffeomorphism $\psi : M_1 \to M_2$ such that

$$\langle v, w \rangle_{g_1} = \langle d\psi|_p v, d\psi|_p w \rangle_{g_2}, \qquad v, w \in T_p M.$$

Example:



From the point of view of Riemannian manifolds, two isometric manifolds are the same.

Toy Model for Earthquakes

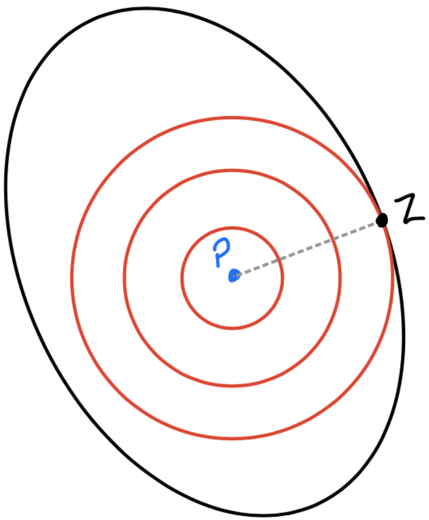
M is a Riemannian manifold with boundary.

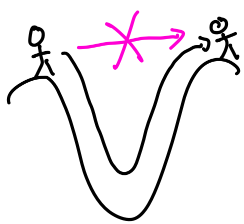
Internal point source wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g)u(x,t) = \delta_p(x)\delta_{t_0}(t), & \text{in } M \times \mathbb{R}, \\ u(x,t) = 0, \quad t < t_0, \ x \in M, & \end{cases}$$

where Δ_g is the Laplace-Beltrami operator of metric g.

t_0 known





Boundary Distance Function

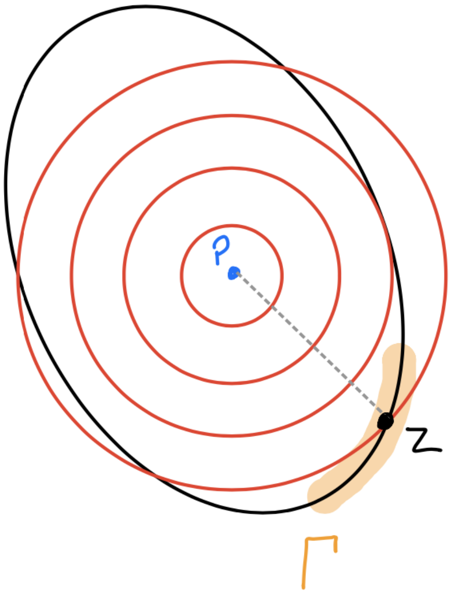
$$r_p : \partial M \to \mathbb{R}, \qquad r_p(z) = d(p,z).$$

Arrival Time

$$\underbrace{T_{p,t_0}(z)}_{\text{measure}} = \underbrace{r_p(z)} + \underbrace{t_0}_{\text{known}}$$

Travel Time Data

Let (M, g) be an n-dimensional Riemannian manifold with smooth boundary ∂M and $\Gamma \subset \partial M$ is the boundary measurement region.



Boundary Distance Function

$$r_p : \partial M \to \mathbb{R}, \qquad r_p(z) = d(p, z).$$

Travel Time Data

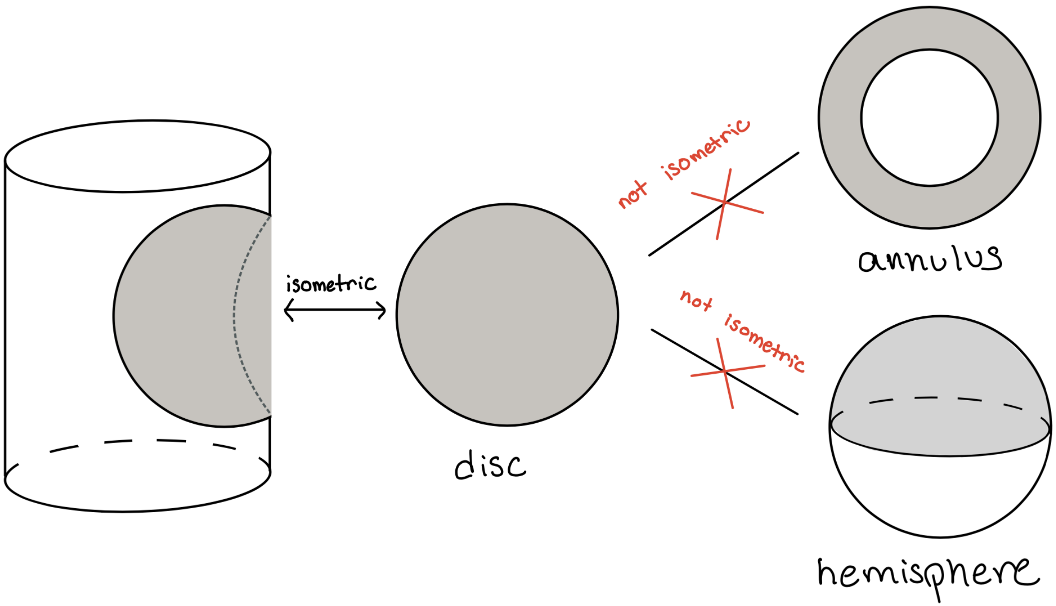
$$\Gamma \quad \text{and} \quad \{r_p|_\Gamma \in C(\Gamma) \ : \ p \in M\}$$

Main Goal

Main Question

Is the travel time data enough to determine the isometry class of (M, g)?

Note: the isometry class is the most we can ask for.



Strictly Convex Boundary Required

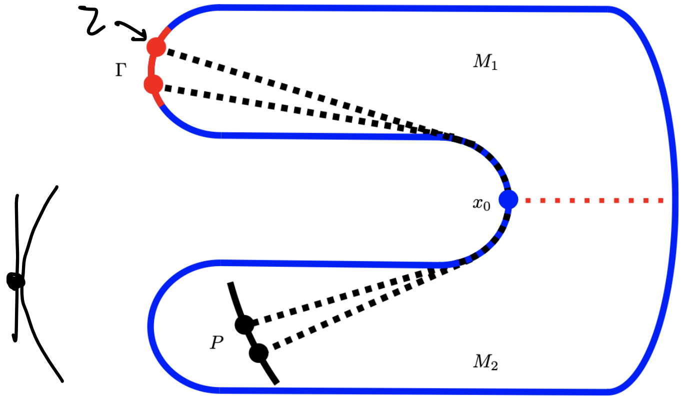
The Second Fundamental Form of the boundary is

$$II(v, w) = \langle n, \nabla_v w \rangle_g n, \qquad v, w \in T_z(\partial M)$$

where ∇ denotes the Levi-Civita connection and $n = n(z)$ is the unit outer normal to the boundary.

The boundary of M is **strictly convex** if the Second Fundamental Form of the boundary is positive-definite for all $z \in \partial M$.

We must exclude the following case:

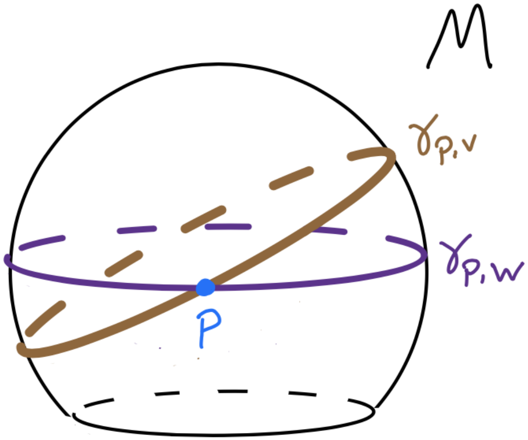


This boundary is non-convex

Strictly convex boundary ensures any two points inside the manifold can be connected with a distance minimizing geodesic whose image is contained in the interior of M.

Non-trapping Required

We must exclude the following case:



This manifold traps geodesics

Define the **Exit time function**:

$$T_{exit} : SM \to \mathbb{R}$$
$$T_{exit}(p, v) = \sup\{t > 0 \ : \ \gamma_{p,v}(t) \in M\}.$$

Impose that the manifold is non-trapping, so that: $T_{exit}(p, v) < \infty$

If the boundary is strictly convex and the manifold is non-trapping then T_{exit} is smooth on $SM \setminus S(\partial M)$.

Main Theorem

Let (M_1, g_1) and (M_2, g_2) be compact, connected, oriented Riemannian manifolds with smooth boundaries ∂M_1 and ∂M_2 and open measurement regions $\Gamma_i \subset \partial M_i$ respectively.

> We say that the travel time data of (M_1, g_1) and (M_2, g_2) **coincide** if there exists a diffeomorphism $\phi : \partial M_1 \to \partial M_2$ such that $\phi(\Gamma_1) = \Gamma_2$ and
> $$\{(r_x \circ \phi^{-1})|_{\Gamma_2} \ : \ x \in M_1\} = \{r_y|_{\Gamma_2} \ : \ y \in M_2\}.$$

Main Theorem

If the travel time data of (M_1, g_1) and (M_2, g_2) coincide, then the Riemannian manifolds M_1 and M_2 are isometric.

''D_1'' D_2''

$$\begin{tikzcd} \Gamma_1 \arrow[rr, bend right, "\phi"] & \{r_x|_{\Gamma_1}\} \arrow[d] & \Gamma_2 \arrow[d] & \{r_y|_{\Gamma_2}\} \\ & M_1 \arrow[r, leftrightarrow] & M_2 & \end{tikzcd}$$

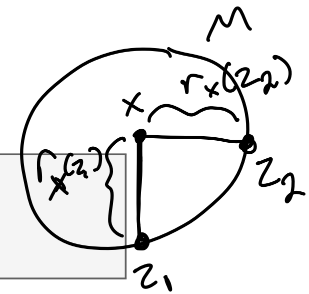
Previous Works

- Travel Time Data

  Key (a) [Kachalov et al., 2001] and [Kurylev, 1997] consider the full-boundary case, when $\Gamma = \partial M$ on a Riemannian manifold.

  (b) [de Hoop et al., 2019] consider the full-boundary case, when $\Gamma = \partial M$ on a compact Finsler manifold.

- Distance Difference Data



Distance Difference Function:

$$D_x(z_1, z_2) = T_{x,t_0}(z_1) - T_{x,t_0}(z_2) = r_x(z_1) - r_x(z_2).$$

  (a) [Lassas and Saksala, 2019] consider M as an open subset of the Riemannian manifold N. The distance difference data is then $N \setminus M$ and $\{D_x \ : \ x \in M\}$.

  (b) [de Hoop and Saksala, 2019] consider when M is a compact Riemannian manifold satisfying certain visibility conditions.

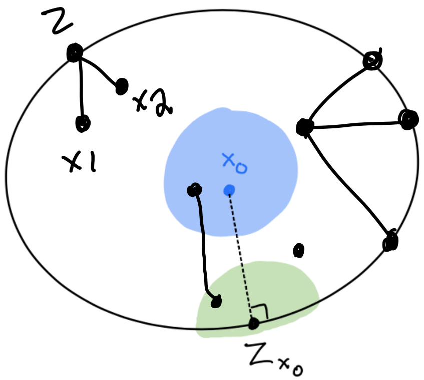
  (c) [Ivanov, 2020a] considers the full-boundary case on a complete Riemannian manifold.

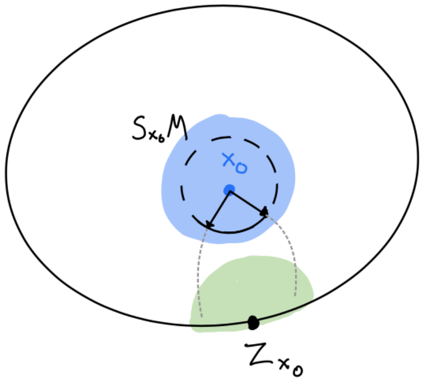
Full Boundary

[Kachalov et al., 2001] considers the full-boundary case, $\Gamma = \partial M$.

Grad $d = \dot{\gamma}$

1. Any point $x_0 \in M$ can be connected to the nearest boundary point z_{x_0} by a geodesic that is normal to the boundary.
2. Then there are neighborhoods $U \subset M$ of x_0 and $V \subset \partial M$ of z_{x_0} with $x \in U$ and $z \in V$ such that
   - (a) $d(x, z) \in C^\infty(U \times V)$
   - (b) The image of $Grad_x d(x, z)|_{x=x_0}$, considered as a function of z, is an open set in $S_{x_0} M$.
3. If $r_{x_1}(z) = r_{x_2}(z)$ for all $z \in \partial M$ then $x_1 = x_2$. This distinguishes points in M.
4. Topological structure from the map $R : x \to r_x$ being an embedding into $C(\partial M)$.
5. Smooth structure from local coordinates for $x_0 \in M$
   - (a) For x near ∂M, $x \mapsto (d(x, \partial M), z_{x_0})$.
   - (b) For $x \in M^{int}$, $x \mapsto (d(x, z_1), \ldots, d(x, z_{n-1}), d(x, z_{x_0}))$.
6. Riemannian structure from metric reconstruction in $S_{x_0} M$.





Our Approach

Our general approach will follow the proof of [Kachalov et al., 2001].

1. Start at the boundary and work inwards.
2. Key Get a smoothness result.
3. Travel time data distinguishes the points in M.
4. Recover topological structure from the embedding R.
5. Recover smooth local coordinates.
6. Recover Riemannian structure.

Start At The Boundary And Work Inwards

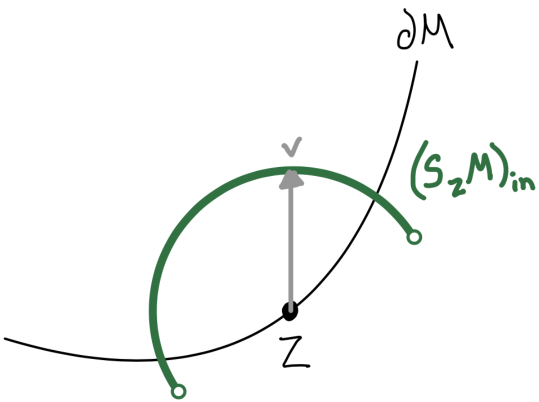
Equivalence of the following sets, for $z \in \Gamma$:

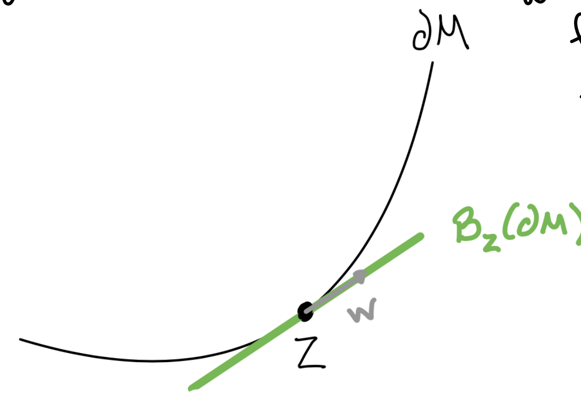
$$(S_z M)_{in} = \{v \in T_z M \ : \ \langle v, n \rangle_g < 0, |v|_g = 1\}, \qquad B_z(\partial M) = \{w \in T_z \partial M \ : \ |w|_g < 1\}$$

"inwards"

equivalent

we can figure this out on Γ





We can find $g|_\Gamma$ so we will want to be working with $B_z(\partial M)$.

(a) We can find the length of any smooth curve in Γ using boundary distance functions.

(b) The lengths of the curves will tell us $g_{ij}|_\Gamma$.

Cut Locus

R.manifold without boundary, N:

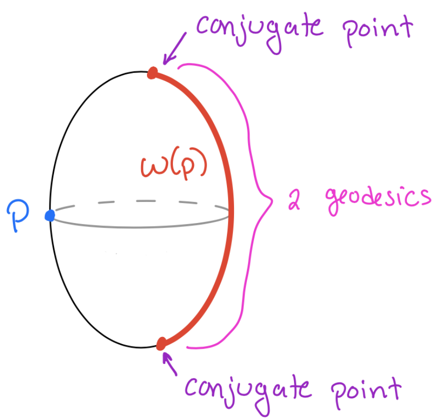
For a point $p \in N$ and $v \in S_pN$,

$$T_{cut}(p, v) = \sup\{t > 0 \ : \ d(p, \gamma_{p,v}(t)) = t\}.$$

$\gamma_{p,v}$ isn't distance minimizing

$q = \gamma_{p,v}(t)$ is a **conjugate point** of p along $\gamma_{p,v}$ if tv is a critical point of exp_p.

$$\omega(p) = \{q \in N \ : q = \gamma_{p,v}(T_{cut}(p, v))\}.$$



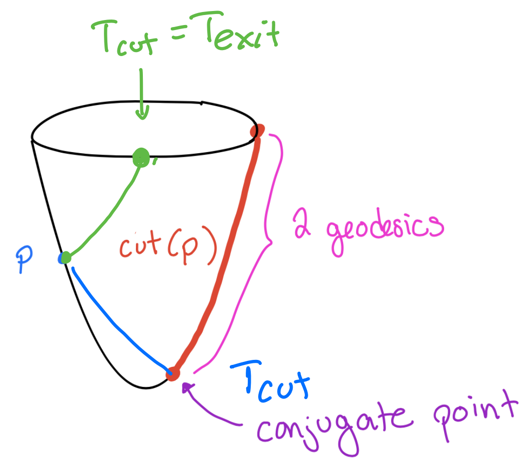
R.manifold with boundary, M:

For a point $p \in M$ and $v \in S_pM$,

$$T_{cut}(p, v) = \sup\{t > 0 \ : \ d(p, \gamma_{p,v}(t)) = t\}.$$

$q = \gamma_{p,v}(t)$ is a **conjugate point** of p along $\gamma_{p,v}$ if tv is a critical point of exp_p.

$$cut(p) = ????$$



Cut Locus (2)

R.manifold without boundary, N:

For a point $p \in N$ and $v \in S_pN$,

$$\omega(p) = \{q \in N \ : \ q = \gamma_{p,v}(T_{cut}(p,v))\}.$$

Properties:

1. Conjugate points are 'symmetric'.
2. If $q \in \omega(p)$ then q is either a conjugate point or there are two distance minimizing geodesics from p to q.
3. $d(\cdot,\cdot)$ is smooth outside of $\omega(p)$.

R.manifold with boundary, M:

For a point $p \in M$ and $v \in S_pM$,

$$cut(p) := \{q \in M \ : q = \gamma_{p,v}(T_{cut}(p,v)), \text{Symmetry}\begin{cases} T_{cut}(p,v) = T_{cut}(q,w), \\ w = -\dot{\gamma}_{p,v}(T_{cut}(p,v))\}. \end{cases}$$

Then:

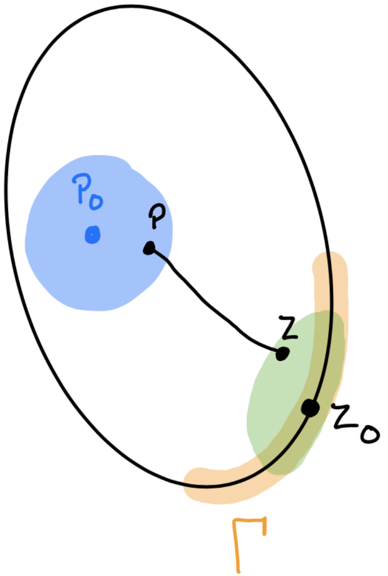
1. Conjugate points are 'symmetric'.
2. If $T_{cut}(p,v) < T_{exit}(p,v)$ and $q \in cut(p)$ then q is either a conjugate point or there are two distance minimizing geodesics from p to q.
3. $d(\cdot,\cdot)$ is smooth $\underbrace{????}_{\text{where}}$

Smoothness

Outside of the cut locus, the distance function should be smooth. Since this has yet to be shown, make the following assumption.

Regularity Assumption

For all $p_0 \in M$ there exists $z_0 \in \Gamma$ such that there are neighborhoods U_{p_0} of p_0 and V_{z_0} of z_0 where $F : U_{p_0} \times V_{z_0} \to \mathbb{R}$ and $F(p, z) = d(p, z) = r_p(z)$ is smooth.



Examples that satisfy all assumptions:

- Unit disc, D^1
- Hemispheres of S^2
- Convex subsets of $\mathbb{R}^n$
- Simple manifolds

Topological Structure

Consider $R : M \to C(\Gamma)$ so that $R : p \mapsto r_p|_\Gamma$.

- Using the Regularity Assumption, we can separate the data.

  For all p_1 and p_2 in M, if $r_{p_1}(z) = r_{p_2}(z)$ for all $z \in \Gamma$ then $p_1 = p_2$.

- This implies the injectivity of R.
- Since the target space has L^∞ norm, it satisfies the Lipschitz inequality

$$\|r_{p_1} - r_{p_2}\|_\infty \leq d(p_1, p_2),$$

  so R is continuous.

- Since M is compact, and R is continuous, then R is a closed map.

Thus, R is a topological embedding.

Smooth Structure

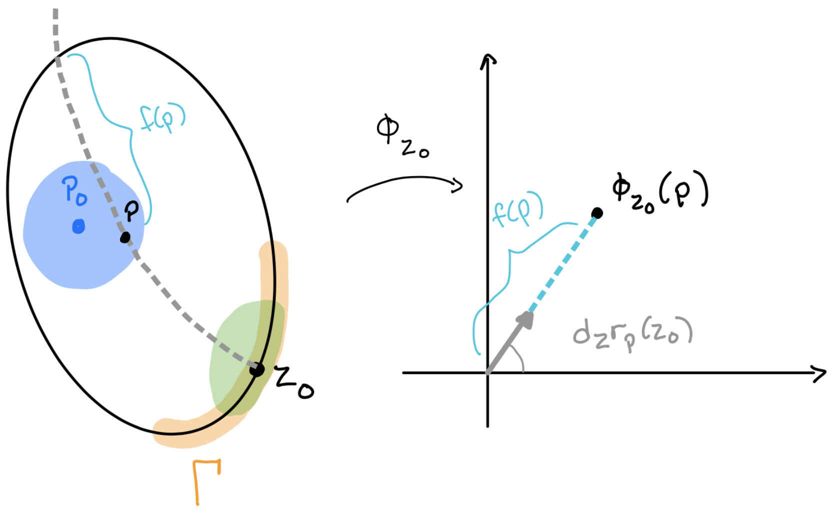
Choose $p_0 \in M$ and $z_0 \in \Gamma$ as in the Regularity Assumption. Make a local coordinate system using a function $\phi_{z_0} : U_{p_0} \to W \subset T_{z_0}M$ such that

$$\phi_{z_0}(p) = (d_z r_p(z_0), \underbrace{T_{exit}(z_0, d_z r_p(z_0)) - r_p(z_0)}_{f(p)}) \tag{1}$$

where $d_z r_p$ is the boundary gradient of r_p. Observe that

$$\phi_{z_0}^{-1}(v) = \exp_{z_0}\left(\left(T_{exit}(z_0, \frac{v}{|v|}) - |v|\right)\frac{v}{|v|}\right). \tag{2}$$

Together, equations (1) and (2) make ϕ_{z_0} a diffeomorphism.

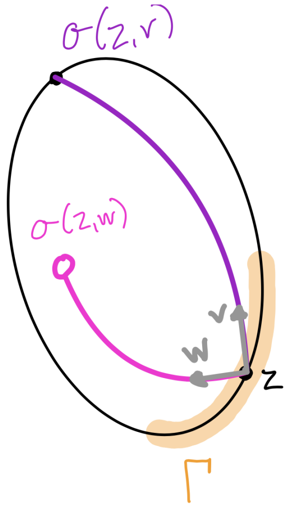


Sigma Sets

From the data we don't know $T_{exit}(z, v)$. For $v \in B_z(\partial M)$,

$$\sigma(z, v) = \{p \in M \mid r_p(z) \text{ is } C^1\text{-smooth in a neighborhood of } z,\ d_z r_p(z) = -v\} \cup \{z\}.$$

[Lassas and Saksala, 2019] show $\sigma(z, v)$ is the trace of geodesic $\gamma_{z,v}$ until the first cut point.



Theorem

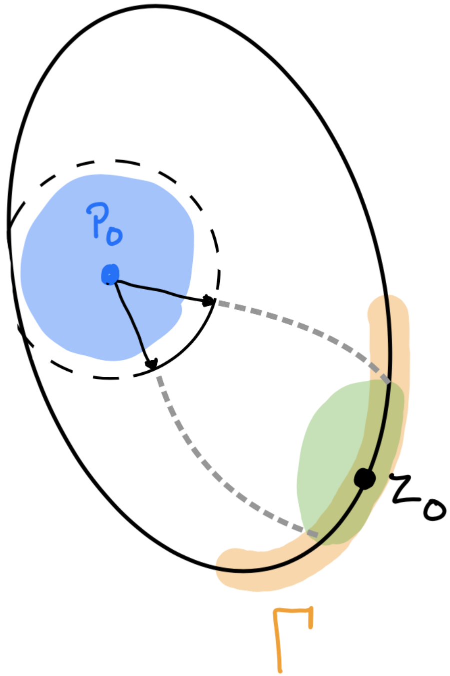
If $\sigma(z, v)$ is closed then $T_{exit}(z, v) = \sup_{x \in \sigma(z,v)} r_x(z)$.

Thus, we can find T_{exit} in a data-driven manner.

Metric Reconstruction

Using the Regularity Assumption, the image of distance functions is an open set in $S_{p_0}M$.

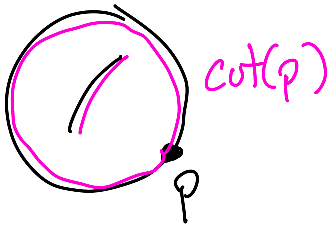
1. Make a basis of n vectors in the open set in $S_{p_0}M$.
2. In this subset there is a norm structure.
3. The norm $\|\cdot\|_g$ of the unit vectors is 1 in $S_{p_0}M$.
4. Using the polarization identity $\langle v, w\rangle_g = \frac{1}{2}(\|v\|_g + \|w\|_g - \|v-w\|_g)$, we have $\langle\cdot,\cdot\rangle_g$.
5. This creates the metric g.



Progress

1. Start at the boundary and work inwards.
   - Define proper notion of cut locus. ← ?
2. ? → Get a smoothness result.
3. Travel time data distinguishes the points in M.
4. Recover topological structure from the embedding R.
5. Recover smooth local coordinates.
6. Recover Riemannian structure.

Inv. prob.



Future Directions

Investigate stability.

- For two different but ‘close’ data sets then their isometry classes are ‘close’.
- Similar results in [Katsuda et al., 2007] and [Ivanov, 2020b] for full-boundary.
- This may require bounds on the diameter, curvature, and injectivity radius of M or Γ.

Consider Finsler manifold

Finale

Thanks everyone!

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Any questions?

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