

CHAPTER 2

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**Prerequisites:**

- Know derivatives of:  $\sin(x)$ ,  $\cos(x)$ ,  $\tan(x)$ ,  $\csc(x)$ ,  $\sec(x)$ ,  $\cot(x)$ ,  $\frac{1}{1+x^2}$
- Know integrals of:  $\sin(ax)$ ,  $\cos(ax)$ ,  $\csc^2(ax)$ ,  $\sec^2(ax)$ ,  $\frac{1}{a^2+x^2}$ , where  $a$  is a constant
- Use U-substitution and Integration by Parts to evaluate integrals
- Given a right-triangle and an angle  $\theta$ , relate the sides using  $\sin(\theta)$ ,  $\cos(\theta)$ , and  $\tan(\theta)$ . (i.e. know SOH CAH TOA)
- Know  $\cos(\theta)$ ,  $\sin(\theta)$ ,  $\tan(\theta)$  when  $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi$
- Memorize trig identities:  $\sin^2(x) + \cos^2(x) = 1$  and  $\tan^2(x) + 1 = \sec^2(x)$  and half-angle formulas.
- Perform polynomial division

**2.1:** Trig. Integrations

- Be able to integrate:  $\int \sin^m(x) \cos^n(x) dx$  for different powers  $m$  and  $n$ .
- Be able to integrate:  $\int \tan^m(x) \sec^n(x) dx$  for different powers  $m$  and  $n$ .

Note: If Case 4 ([ $m$  is even and  $n$  is odd] OR [ $n = 0$ ]) is on the Exam, you will be given a hint on how to solve it.

**2.2:** Trig. Substitutions

- Be able to integrate functions with  $\sqrt{a^2 - u^2}$ ,  $\sqrt{u^2 - a^2}$ , and  $\sqrt{u^2 + a^2}$ . Note, showing full work for this question will entail:
  - Drawing the substitution right-triangle
  - Use SOH CAH TOA to perform the substitution
  - Evaluate the integral using the techniques in Section 2.1

**2.3:** Partial Fractions

- Be able to integrate ratios of polynomials,  $\frac{P(x)}{Q(x)}$   
Roughly speaking, the process is:
  - If  $(\text{degree of } P(x)) \geq (\text{degree of } Q(x))$ , do polynomial division
  - Factor  $Q(x)$  to determine its linear and/or quadratic terms
  - Use the 4 cases discussed in class to decompose the fraction.
  - Determine the constants (you will likely need to solve a system of equations)
  - Integrate

**2.4:** Table of Integrals will **NOT** be on the Exam

(Continued on next page)

**2.5:** Numerical Integration

- Approximate an integral using the Midpoint Rule, Trapezoid Rule, or Simpson's Rule
- Understand how the Trapezoidal Rule and Simpson's Rule were derived (main ideas, you do not need to memorize each step).
- **You will be given the equations for the error bound**, you must know how to use them.
- Compare/contrast the error of the estimates and the upper bound for the absolute value of the error.

**2.6:** Improper Integrals

- Be able to integrate Type 1 Integrals:  $\int_a^\infty f(x) dx$ ,  $\int_{-\infty}^b f(x) dx$ ,  $\int_{-\infty}^\infty f(x) dx$
- Recognize when a function is discontinuous or has an asymptote
- Be able to integrate Type 2 Integrals:  $\int_a^b f(x) dx$  with discontinuities at  $a$ ,  $b$ , or  $a < c < b$
- Know the definitions of convergent and divergent indefinite integrals.

EXERCISES: (Note: the level of difficulty of the \*Challenge\* questions will not be on your test)

1.  $\int \sin^6(x) \cos^3(x) \, dx$

2.  $\int \frac{\cos(x)}{\sin(x)} \, dx$

3.  $\int \tan^3(x) \sec^4(x) \, dx$

4.  $\int \frac{\sec^4(\ln(x))}{x} \, dx$

5. (Challenge)  $\int \tan^2(x) \, dx$

6. (Challenge)  $\int \sec^3(x) \, dx$

7. Evaluate the following integrals using trigonometric substitution:

(a)  $\int \frac{1}{x^2\sqrt{8+2x^2}} \, dx$

(d)  $\int_0^{\sqrt{3}} \frac{1}{(4-t^2)^{5/2}} \, dt$

(b)  $\int x^3 \sqrt{9-x^2} \, dx$

(e)  $\int \frac{x}{\sqrt{x^2+4x+8}} \, dx$

(c)  $\int \frac{dx}{(4x^2+9)^2}$

(f)  $\int e^x \sqrt{1-e^{2x}} \, dx$

8. (Challenge) Find the arclength of  $y = \ln(x)$  over the interval  $[1, 5]$

9. (Challenge)  $\int \frac{1}{x^2+a^2} \, dx$  for  $a > 0$ .

10. Evaluate the following integrals using partial fraction decomposition:

(a)  $\int \frac{7x+12}{x^2(x+4)} \, dx$

(c)  $\int \frac{1}{(x+1)^2(x^2+1)} \, dx$

(b)  $\int \frac{16x^3+11x^2+68x+22}{(x^2+2)(x^2+5)} \, dx$

(d) (Challenge)  $\int \frac{6x^2+19x-4}{(x-1)^2(x+6)} \, dx$

11. Write out the form of the partial fraction decomposition but do NOT determine the numerical values of the coefficients:

(a)  $\int \frac{2x}{(x+3)(3x+1)} \, dx$

(d)  $\int \frac{x-7}{(x-4)^2(2x^2+1)^2} \, dx$

(b)  $\int \frac{1}{x^3+2x^2+x} \, dx$

(e)  $\int \frac{6}{x^2(x-1)(4x^2+3)} \, dx$

(c)  $\int \frac{x^2-x+6}{x^3+3x} \, dx$

(f)  $\int \frac{x^4}{(x^3+x)(x^2-4x+3)} \, dx$

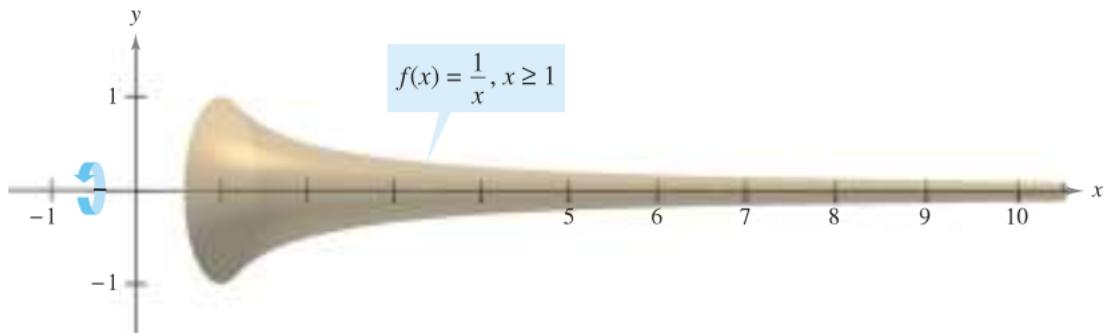
12. Approximate  $\int_0^1 \cos(x^2) dx$  with  $n = 4$ :
- Using the Midpoint Rule
  - Using the Trapezoid Rule
  - Using Simpson's Rule
  - Assume that  $\int_0^1 \cos(x^2) dx = 0.904524$ . Which approximation method is the most accurate? Why?
13. Summarize the meaning behind the expression:  $|E_T| \leq \frac{k(b-a)^3}{12n^2}$  using your own words.
14. Approximate the definite integral using the trapezoidal rule for  $n = 8$
- $\int_0^1 x\sqrt{4-x^2} dx$
  - Find the upper bound for the absolute value of the error
  - Find the exact value of the definite integral
15. Approximate the definite integral using Simpson's rule for  $n = 8$
- $\int_0^4 x^4 dx$
  - Find the upper bound for the absolute value of the error
  - Find the exact value of the definite integral
16. Approximate the definite integral using Simpson's rule for  $n = 4$
- $\int_1^9 \ln(1+x) dx$
  - Find the upper bound for the absolute value of the error
17. **True or False:** To use Trapezoidal Rule,  $n$  must be an even number.
18. **True or False:** To use Simpson's Rule,  $n$  must be an even number.
19. (Challenge) For  $\int_1^2 \frac{1}{x} dx$ , what should  $n$  be so that  $|E_S| < 0.0001$ ?
20. Determine if the following integrals are convergent or divergent. Evaluate the integral if it is convergent.
- |  |   |
|--|---|
| <ol style="list-style-type: none"> <li><math>\int_0^\infty xe^{-x} dx</math></li> <li><math>\int_{-5}^1 \frac{1}{10+2x} dx</math></li> </ol> | <ol style="list-style-type: none"> <li><math>\int_2^\infty \frac{9}{(6-3x)^4} dx</math></li> <li><math>\int_0^\infty \sin(x) dx</math></li> </ol> |
|--|---|
21. If it is possible, find the area bounded by  $y = \frac{1}{\sqrt[3]{x}}$ , the  $x$ -axis, the  $y$ -axis, and the line  $x = 4$ .

22. What is wrong with the following calculation?

$$\int_0^3 \frac{1}{(x-2)^2} dx = -\frac{1}{x-2} \Big|_0^3 = -1 + \frac{1}{2} = -\frac{1}{2}$$

23. Explain why  $\int_{-1}^1 \frac{1}{x^3} dx \neq 0$ .

24. (Challenge/ Test #2 Bonus ) The solid formed by revolving (about the x-axis) the unbounded region lying between the graph of  $f(x) = \frac{1}{x}$ , the  $x$ -axis, and  $x \geq 1$  is called **Gabriel's Horn**. Show that the volume of this solid converges, but it has an infinite surface area.



$$\begin{aligned}
 ① \int \sin^6(x) \cos^3(x) dx &= \int \sin^6(x) \cos^2(x) \cos(x) dx \\
 &= \int \sin^6(x) (1 - \sin^2 x) \cos x dx \\
 &\quad u = \sin(x) \\
 &\quad du = \cos x dx \\
 &= \int u^6 (1 - u^2) du \\
 &= \int u^6 - u^8 du \\
 &= \frac{u^7}{7} - \frac{u^9}{9} + C \quad = \boxed{\frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C}
 \end{aligned}$$

$$\begin{aligned}
 ② \int \frac{\cos x}{\sin x} dx &= \int \frac{1}{u} du = \ln|u| + C = \boxed{\ln|\sin x| + C} \\
 &\quad u = \sin x \\
 &\quad du = \cos x dx
 \end{aligned}$$

$$\begin{aligned}
 ③ \int \tan^3 x \sec^4 x dx &= \int \tan^2 x \sec^3 x (\sec x \tan x dx) \\
 &= \int (\sec^2 x - 1) \sec^3 x (\sec x \tan x dx) \\
 &\quad u = \sec x \\
 &\quad du = \sec x \tan x dx \\
 &= \int (u^2 - 1) u^3 du \\
 &= \int u^5 - u^3 du \\
 &= \frac{u^6}{6} - \frac{u^4}{4} + C = \boxed{\frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C}
 \end{aligned}$$

$$\begin{aligned}
 ④ \int \frac{\sec^4(\ln(x))}{x} dx &= \int \sec^4(u) du = \int \sec^2 u (\sec^2 u du) \\
 &\quad u = \ln(x) \\
 &\quad du = \frac{1}{x} dx \\
 &= \int (\tan^2 u + 1) (\sec^2 u du) \\
 &\quad \tilde{u} = \tan u \\
 &\quad d\tilde{u} = \sec^2 u du \\
 &= \int \tilde{u}^2 + 1 d\tilde{u} \\
 &= \frac{\tilde{u}^3}{3} + \tilde{u} + C
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} \tan^3 u + \tan u + C \\
 &= \boxed{\frac{1}{3} \tan^3(\ln(x)) + \tan(\ln(x)) + C}
 \end{aligned}$$

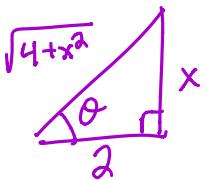
$$\begin{aligned}
 ⑤ \int \tan^2(x) dx &= \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} - 1 dx \\
 &= \int \sec^2 x - 1 dx \\
 &= \boxed{\tan x - x + C}
 \end{aligned}$$

$$\begin{aligned}
 ⑥ \int \sec^3 x dx &= \int \sec^2 x \sec x dx \\
 u = \sec x \quad dv &= \sec^2 x dx \\
 du = \sec x \tan x dx \quad v &= \tan x \\
 &= \sec x \tan x - \int \tan x (\sec x \tan x dx) \\
 &= \sec x \tan x - \int \tan^2 x \sec x dx \\
 &= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
 \underbrace{\int \sec^3 x dx} &= \sec x \tan x - \underbrace{\int \sec^3 x dx}_{\text{blue wavy line}} + \int \sec x dx
 \end{aligned}$$

$$\begin{aligned}
 2 \underbrace{\int \sec^3 x dx} &= \sec x \tan x + \int \sec x dx \\
 &= \sec x \tan x + \ln |\sec x + \tan x| + C
 \end{aligned}$$

$$\int \sec^3 x dx = \boxed{\frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C}$$

$$7a \int \frac{1}{x^2 \sqrt{8+2x^2}} dx = \int \frac{1}{x^2 \sqrt{2(4+x^2)}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{x^2 \sqrt{4+x^2}} dx$$



$x = 2 \tan \theta$   
 $dx = 2 \sec^2 \theta d\theta$   
 $\sqrt{4+x^2} = 2 \sec \theta$

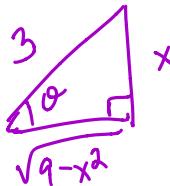
$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \int \frac{1}{(2\tan\theta)^2 \cancel{2\sec\theta}} \cancel{2\sec^2\theta} d\theta \\
 &= \frac{1}{\sqrt{2}} \int \frac{1}{4\tan^2\theta} \sec\theta d\theta \\
 &= \frac{1}{4\sqrt{2}} \int \frac{\cos^2\theta}{\sin^2\theta} \frac{1}{\cos\theta} d\theta \\
 &= \frac{1}{4\sqrt{2}} \int \frac{\cos\theta}{\sin^2\theta} d\theta
 \end{aligned}$$

$$u = \sin \theta$$

$$du = \cos\theta d\theta$$

$$\begin{aligned}
 &= \frac{1}{4\sqrt{2}} \int \frac{1}{u^2} du \\
 &= \frac{1}{4\sqrt{2}} \left( -\frac{1}{u} \right) + C \\
 &= -\frac{1}{4\sqrt{2}} \frac{1}{\sin\theta} + C \quad \text{since } \sin\theta = \frac{x}{\sqrt{4+x^2}} \\
 &= -\frac{1}{4\sqrt{2}} \frac{\sqrt{4+x^2}}{x} + C
 \end{aligned}$$

$$7b \int x^3 \sqrt{9-x^2} dx$$



$x = 3 \sin \theta$   
 $dx = 3 \cos\theta d\theta$   
 $\sqrt{9-x^2} = 3 \cos\theta$

$$= \int (3\sin\theta)^3 3\cos\theta (3\cos\theta d\theta) = \int 3^5 \sin^3\theta \cos^2\theta d\theta$$

$$= 3^5 \int \sin^2\theta \cos^3\theta (\sin\theta d\theta) = 3^5 \int (1-\cos^2\theta) \cos^3\theta (\sin\theta d\theta)$$

$$u = \cos \theta$$

$$du = -\sin \theta d\theta$$

$$= -3^5 \int (1-u^2) u^2 du$$

$$= -3^5 \int u^2 - u^4 du$$

$$= -3^5 \left( \frac{u^3}{3} - \frac{u^5}{5} \right) + C$$

$$= -3^5 \left( \frac{1}{3} \cos^3 \theta - \frac{1}{5} \cos^5 \theta \right) + C$$

$$= \boxed{-3^5 \left( \frac{1}{3} \left( \frac{\sqrt{9-x^2}}{3} \right)^3 - \frac{1}{5} \left( \frac{\sqrt{9-x^2}}{3} \right)^5 \right) + C}$$

(7c)  $\int \frac{1}{(4x^2+9)^2} dx = \int \frac{1}{(\sqrt{4x^2+9})^4} dx$

$$x = \frac{3}{2} \tan \theta$$

$$dx = \frac{3}{2} \sec^2 \theta d\theta$$

$$\sqrt{4x^2+9} = 3 \sec \theta d\theta$$

$$= \int \frac{1}{(3 \sec \theta)^4} \cdot \frac{3}{2} \sec^2 \theta d\theta = \frac{\frac{3}{2}}{3^6} \int \frac{1}{\sec^2 \theta} d\theta$$

$$= \frac{\frac{3}{2}}{3^6} \int \cos^2 \theta d\theta$$

$$= \frac{\frac{3}{2}}{3^6} \int \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= \frac{\frac{3}{2}}{3^6} \left( \int \frac{1}{2} d\theta + \frac{1}{2} \int \cos(2\theta) d\theta \right)$$

$$= \frac{\frac{3}{2}}{3^6} \left( \frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) \right) + C$$

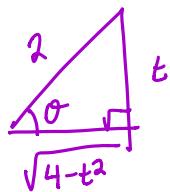
since  $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$

$$= \frac{\frac{3}{2}}{3^6} \left( \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta \right) + C$$

$$= \frac{\frac{3}{2}}{3^6} \left( \frac{1}{2} \arctan \left( \frac{2x}{3} \right) + \frac{1}{2} \frac{2x}{\sqrt{4x^2+9}} \frac{3}{\sqrt{4x^2+9}} \right) + C$$

7d

$$\int_0^1 \frac{1}{(4-t^2)^{5/2}} dt = \int_0^{\sqrt{3}} \frac{1}{(\sqrt{4-t^2})^5} dt$$



$$\begin{aligned} t &= 2 \sin \theta \\ dt &= 2 \cos \theta d\theta \\ \sqrt{4-t^2} &= 2 \cos \theta \end{aligned}$$

when  $t=1$   
then  $1=2 \sin \theta$   
so  $\theta=\arcsin(\frac{1}{2})=\frac{\pi}{6}$

when  $t=0$   
then  $0=2 \sin \theta$   
so  $\theta=\arcsin(0)=0$

$$= \int_0^{\pi/6} \frac{1}{(2 \cos \theta)^5} 2 \cos \theta d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2^4} \frac{1}{\cos^4 \theta} d\theta = \int_0^{\pi/6} \frac{1}{2^4} \sec^4 \theta d\theta$$

$$\begin{aligned} &= \frac{1}{2^4} \int_0^{\pi/6} \sec^2 \theta (\sec^2 \theta d\theta) \\ &= \frac{1}{2^4} \int_0^{\pi/6} (\tan^2 \theta + 1) (\sec^2 \theta d\theta) \end{aligned}$$

when  $\theta=\frac{\pi}{6}$   
then  $u=\tan(\frac{\pi}{6})=\frac{1}{\sqrt{3}}$   
when  $\theta=0$   
then  $u=\tan(0)=0$

$$= \frac{1}{2^4} \int_0^{\sqrt{3}} u^2 + 1 du$$

$$= \frac{1}{2^4} \left[ \frac{u^3}{3} + u \right]_0^{\sqrt{3}}$$

$$= \boxed{\frac{1}{2^4} \left( \frac{1}{3} \left( \frac{1}{\sqrt{3}} \right)^3 + \frac{1}{\sqrt{3}} \right)}$$

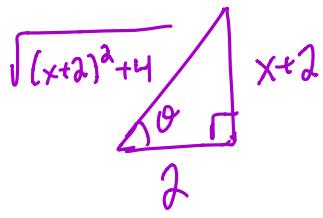
7e

$$\int \frac{x}{\sqrt{x^2+4x+8}} dx$$

↪ complete the square

$$x^2+4x+8 = (x^2+4x+4) - 4 + 8 = (x+2)^2 + 4$$

$$= \int \frac{x}{\sqrt{(x+2)^2 + 4}} dx$$



$$x = 2 \tan \theta - 2$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\sqrt{(x+2)^2 + 4} = 2 \sec \theta$$

$$= \int \frac{2 \tan \theta - 2}{2 \sec \theta} 2 \sec^2 \theta d\theta = \int 2 \tan \theta \sec \theta - 2 \sec \theta d\theta$$

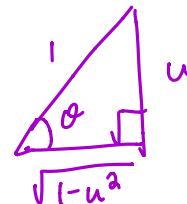
$$= 2 \sec \theta - 2 \ln |\sec \theta + \tan \theta| + C$$

$$= \boxed{2 \left( \frac{\sqrt{(x+2)^2 + 4}}{2} \right) - 2 \ln \left| \frac{\sqrt{(x+2)^2 + 4}}{2} + \frac{x+2}{2} \right| + C}$$

$$(7f) \int e^x \sqrt{1-e^{2x}} dx = \int \sqrt{1-u^2} du$$

$$u = e^x$$

$$du = e^x dx$$



$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$\sqrt{1-u^2} = \cos \theta$$

$$= \int \cos \theta (\cos \theta d\theta)$$

$$= \int \cos^2 \theta d\theta$$

$$= \int \frac{1}{2} + \frac{1}{2} \cos(2\theta) d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4} \underbrace{\sin(2\theta)}_{\sin(2\theta) = 2 \sin \theta \cos \theta} + C$$

$$= \frac{1}{2}\theta + \frac{1}{2} \sin \theta \cos \theta + C$$

$$= \frac{1}{2} \arcsin(u) + \frac{1}{2} u \sqrt{1-u^2} + C$$

$$= \boxed{\frac{1}{2} \arcsin(e^x) + \frac{1}{2} e^x \sqrt{1-e^{-2x}} + C}$$

$$\textcircled{8} \quad L = \left( -\ln \left( \frac{\sqrt{26}}{5} + \frac{1}{5} \right) + \sqrt{26} \right) - \left( -\ln(\sqrt{2}+1) + \sqrt{2} \right)$$

$$\textcircled{9} \quad \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C$$

$$\textcircled{10a} \quad \int \frac{7x+12}{x^2(x+4)} dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$7x+12 = A(x)(x+4) + B(x+4) + Cx^2$$

$$= Ax^2 + 4Ax + Bx + 4B + Cx^2$$

$$= (A+C)x^2 + (4A+B)x + 4B$$

$$\begin{array}{l} A+C=0 \\ 4A+B=7 \\ 4B=12 \end{array} \xrightarrow{\quad} \begin{array}{l} 4A+3=7 \\ 4A=4 \\ A=1 \end{array} \quad \boxed{C=-1}$$

$$\begin{aligned} \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4} &= \int \frac{1}{x} + \frac{3}{x^2} + \frac{-1}{x+4} dx \\ &= \boxed{\ln|x| - \frac{3}{x} - \ln|x+4| + C} \end{aligned}$$

10b

$$\int \frac{16x^3 + 11x^2 + 68x + 22}{(x^2+2)(x^2+5)} dx = \int \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+5}$$

$$\begin{aligned} 16x^3 + 11x^2 + 68x + 22 &= (Ax+B)(x^2+5) + (Cx+D)(x^2+2) \\ &= Ax^3 + 5Ax + Bx^2 + 5B + Cx^3 + 2Cx^2 + 2D \\ &= (A+C)x^3 + (B+D)x^2 + (5A+2C)x + (5B+2D) \end{aligned}$$

$$\begin{array}{l} \begin{array}{l} 16 = A+C \\ 11 = B+D \\ 68 = 5A+2C \\ 22 = 5B+2D \end{array} \quad \begin{array}{l} 16 = A+C \\ -32 = -2A-2C \\ + (68 = 5A+2C) \\ \hline 36 = 3A \end{array} \quad \boxed{A = 12} \\ \begin{array}{l} 16 = 12+C \\ 4 = C \end{array} \\ \begin{array}{l} 11 = B+D \\ -22 = -2B-2D \\ + (22 = 5B+2D) \\ \hline 0 = 3B \end{array} \quad \boxed{B = 0} \\ \begin{array}{l} 11 = G+D \\ 11 = D \end{array} \end{array}$$

$$\begin{aligned} \int \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+5} &= \int \frac{12x}{x^2+2} + \frac{4x+11}{x^2+5} \\ &= \int \frac{12x}{x^2+2} + \int \frac{4x}{x^2+5} + \int \frac{11}{x^2+5} \end{aligned}$$

$$= \boxed{6 \ln|x^2+2| + 2 \ln|x^2+5| + \frac{11}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) + C}$$

10c

$$\int \frac{1}{(x+1)^2(x^2+1)} dx = \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} dx$$

$$\begin{aligned} 1 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \\ &= Ax^3 + Ax^2 + A + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D \\ &= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D) \end{aligned}$$

$$\begin{array}{l} A+C=0 \\ A+B+2C+D=0 \\ A+C+2D=0 \\ A+B+D=1 \end{array} \quad \begin{array}{rcl} \rightarrow & A+C & = 0 \\ \rightarrow & -(A+C+2D=0) & \\ & -2D & = 0 \end{array} \quad \text{so } \boxed{D=0}$$

$$\begin{array}{rcl} & A+B+2C+D=0 & \\ \downarrow & -(A+B+D=1) & \\ & 2C & = -1 \end{array} \quad \text{so } \boxed{C=-\frac{1}{2}}$$

$$\begin{array}{l} \text{since } A+C=0 \\ A+(-\frac{1}{2})=0 \end{array} \quad \text{so } \boxed{A=\frac{1}{2}}$$

$$\begin{array}{l} \text{since } A+B+D=0 \\ (\frac{1}{2})+B+0=0 \end{array} \quad \text{so } \boxed{B=-\frac{1}{2}}$$

$$\begin{aligned} \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} dx &= \int \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{(x+1)^2} - \frac{\frac{1}{2}x}{x^2+1} \\ &= \boxed{\frac{1}{2} \ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4} \ln|x^2+1| + C} \end{aligned}$$

16d

$$\int \frac{6x^2 + 19x - 4}{(x-1)^2(x+6)} = \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+6}$$

$$\begin{aligned} 6x^2 + 19x - 4 &= A(x-1)(x+6) + B(x+6) + C(x-1)^2 \\ &= Ax^2 + 5Ax - 6A + Bx + 6B + Cx^2 - 2Cx + C \\ &= (A+C)x^2 + (5A+B-2C)x + (-6A+6B+C) \end{aligned}$$

$$\begin{array}{l} A+C=6 \\ 5A+B-2C=19 \\ -6A+6B+C=-4 \end{array} \quad \begin{array}{l} C=6-A \\ B=19-5A+2C=19-5A+2(6-A)=31-7A \\ -6A+6(31-7A)+(6-A)=-4 \\ 192-49A=-4 \\ -49A=-196 \end{array}$$

$$A = 4$$

$$C = 6-4 = 2$$

$$C = 2$$

$$B = 31 - 7(4) = 3$$

$$B = 3$$

$$\int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+6} = \int \frac{4}{x-1} + \frac{3}{(x-1)^2} + \frac{2}{(x+6)} dx$$

$$= \boxed{4 \ln|x-1| - 3 \frac{1}{x-1} + 2 \ln|x+6| + C}$$

II) a)  $\int \frac{2x}{(x+3)(3x+1)} dx = \int \frac{A}{x+3} + \frac{B}{3x+1} dx$

b)  $\int \frac{1}{x^3+2x^2+x} dx = \int \frac{1}{x(x+1)^2} dx = \int \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} dx$

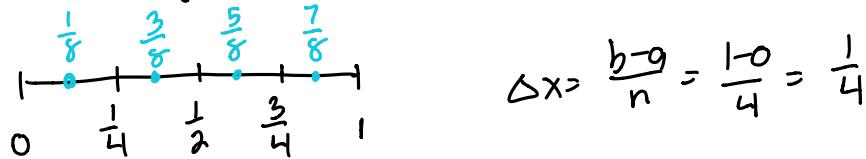
c)  $\int \frac{x^2-x+6}{x^3+3x} dx = \int \frac{x^2-x+6}{x(x^2+3)} dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+3} dx$

d)  $\int \frac{x-7}{(x-4)^2(2x^2+1)^2} dx = \int \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{(2x^2+1)} + \frac{Ex+F}{(2x^2+1)^2} dx$

e)  $\int \frac{6}{x^2(x-1)(4x^2+3)} dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{4x^2+3}$

f)  $\int \frac{x^4}{(x^3+x)(x^2-4x+3)} dx = \int \frac{x^4}{x(x^2+1)(x-3)(x-1)} dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{D}{x-3} + \frac{E}{x-1}$

(12) Approximate  $\int_0^1 \cos(x^2) dx$  with  $n=4$



$$a) M_4 = \frac{1}{4} \left[ f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right) \right]$$

$$= \boxed{\frac{1}{4} \left[ \cos\left(\frac{1}{64}\right) + \cos\left(\frac{9}{64}\right) + \cos\left(\frac{25}{64}\right) + \cos\left(\frac{49}{64}\right) \right]}$$

$$= 0.90891$$

$$b) T_4 = \frac{\left(\frac{1}{4}\right)}{2} \left[ f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1) \right]$$

$$= \boxed{\frac{1}{8} \left[ \cos(0) + 2\cos\left(\frac{1}{16}\right) + 2\cos\left(\frac{1}{4}\right) + 2\cos\left(\frac{9}{16}\right) + 2\cos(1) \right]}$$

$$= 0.89576$$

$$c) S_4 = \frac{\left(\frac{1}{4}\right)}{3} \left[ f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right]$$

$$= \boxed{\frac{1}{12} \left[ \cos(0) + 4\cos\left(\frac{1}{16}\right) + 2\cos\left(\frac{1}{4}\right) + 4\cos\left(\frac{9}{16}\right) + \cos(1) \right]}$$

$$= 0.904501$$

$$d) \text{ Since } \int_0^1 \cos(x^2) dx = 0.904524$$

$$E_M = 0.904524 - 0.90891 = -0.004386$$

$$E_T = 0.904524 - 0.89576 = 0.008764$$

$$E_S = 0.904524 - 0.904501 = 0.000023$$

Simpson's Rule is the most accurate

(13) The expression  $|E_T| \leq \frac{K(b-a)^3}{12n^2}$  lets us create an interval that the error,  $E_T$ , is guaranteed to be in.



The number  $\frac{K(b-a)^3}{12n^2}$  is the "worst possible" error we could have

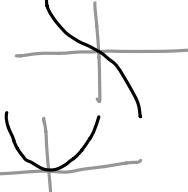
$$\begin{aligned}
 (14) \quad a) T_8 &= \frac{\left(\frac{1}{8}\right)}{2} \left( f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{3}{8}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{5}{8}\right) + 2f\left(\frac{3}{4}\right) \right. \\
 &\quad \left. + 2f\left(\frac{7}{8}\right) + f(1) \right) \\
 &= \frac{1}{16} \left( 0 + \frac{1}{4} \sqrt{4-\frac{1}{64}} + \frac{1}{2} \sqrt{4-\frac{1}{16}} + \frac{3}{4} \sqrt{4-\frac{9}{64}} + \sqrt{4-\frac{1}{4}} + \frac{5}{4} \sqrt{4-\frac{25}{64}} + \frac{3}{2} \sqrt{4-\frac{9}{16}} \right. \\
 &\quad \left. + \frac{7}{4} \sqrt{4-\frac{49}{64}} + \sqrt{4-1} \right) \\
 &= 0.9335...
 \end{aligned}$$

$$\begin{aligned}
 b) \quad f(x) &= x\sqrt{4-x} \\
 f'(x) &= \sqrt{4-x} - x \cdot \frac{1}{2}(4-x)^{-1/2} = (4-x)^{-1/2} \left( 4 - \frac{3}{2}x \right)
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= (4-x)^{-1/2} \left( -\frac{3}{2} \right) + \left( 4 - \frac{3}{2}x \right) \left( -\frac{1}{2} \right) (4-x)^{-3/2} (-1) \\
 &= (4-x)^{-3/2} (2x)(x^2 - 6)
 \end{aligned}$$

$$= \frac{2x(x-6)}{(4-x)^{3/2}}$$

the graph looks like



so  $|f''|$  looks like

$$\text{this means } |f''(x)| \leq |f''(1)| = \left| \frac{-10}{(3)^{3/2}} \right| = \frac{10}{\sqrt{27}}$$

$$\left| E_T \right| \leq \frac{\left( \frac{10}{\sqrt{87}} \right) (1-0)^3}{12 \cdot 8^2} = 0.002505 \dots$$

c)  $\int_0^1 x \sqrt{4-x^2} dx$

You could use u-sub. or trig. sub.!

$$\begin{aligned} u &= 4-x^2 \\ du &= -2x dx \\ -\frac{1}{2} du &= x dx \end{aligned}$$

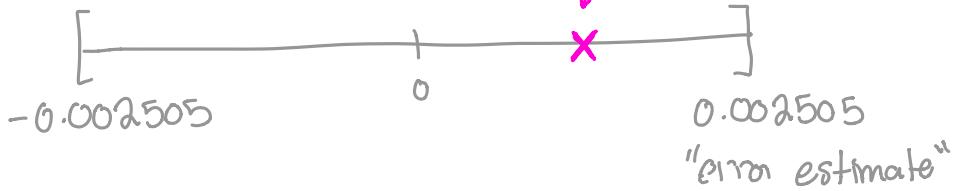
$$\begin{array}{l|l} \text{when } x=1 & u=4-1^2=3 \\ & u=4 \\ \text{when } x=0 & u=4-0=4 \end{array}$$

$$\begin{aligned} &= \int_4^3 -\frac{1}{2} \sqrt{u} du \\ &= -\frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_4^3 \\ &= \left( -\frac{1}{3} (3^{3/2}) \right) - \left( -\frac{1}{3} (4^{3/2}) \right) = \boxed{\left( -3^{1/2} \right) + \frac{8}{3}} \\ &= 0.9346 \end{aligned}$$

Note:  $E_T = 0.9346 - 0.9335$

$$= 0.0011$$

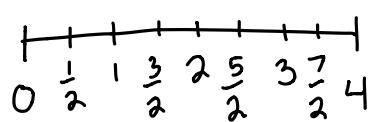
$$E_T = 0.0011$$



0.002505

"error estimate"

15



$$\Delta x = \frac{4-0}{8} = \frac{1}{2}$$

a)  $S_8 = \frac{\left(\frac{1}{2}\right)}{3} \left[ f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4) \right]$

$$= \frac{1}{6} \left[ 0 + 4\left(\frac{1}{2}\right)^4 + 2(1)^4 + 4\left(\frac{3}{2}\right)^4 + 2(2)^4 + 4\left(\frac{5}{2}\right)^4 + 2(3)^4 + 4\left(\frac{7}{2}\right)^4 + 4^4 \right]$$

$$= \boxed{\frac{1}{6} \left[ \frac{1}{4} + 2 + \frac{81}{4} + 32 + \frac{625}{4} + 162 + \frac{2401}{4} + 256 \right]}$$

$$= 204.83\bar{3}$$

b)  $f(x) = x^4$

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f'''(x) = 24x$$

$$f''''(x) = 24 \rightarrow \text{this looks like } \begin{array}{c} 24 \\ \hline - \\ \hline \end{array}$$

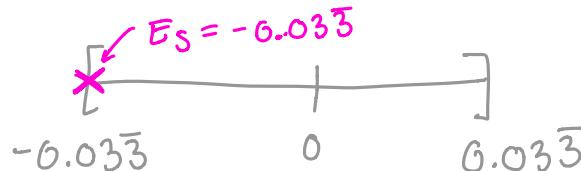
the biggest it can get is 24

$$\text{so } \tilde{k} = 24$$

$$\boxed{|E_S| \leq \frac{24(4-0)^5}{180(8^4)}} = 0.03\bar{3}$$

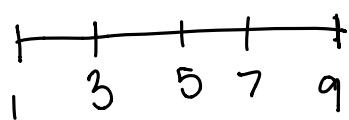
c)  $\int_0^4 x^4 dx = \frac{1}{5} x^5 \Big|_0^4 = \boxed{\frac{1}{5}(4^5)} = 204.8$

Notice:  $E_S = 204.8 - 204.83\bar{3} = -0.03\bar{3}$



but, hey it's in the interval!

16



$$\Delta x = \frac{9-1}{4} = 2$$

$$a) S_4 = \frac{2}{3} [f(1) + 4f(3) + 2f(5) + 4f(7) + f(9)]$$

$$= \boxed{\frac{2}{3} [\ln(2) + 4\ln(4) + 2\ln(6) + 4\ln(8) + \ln(10)]}$$

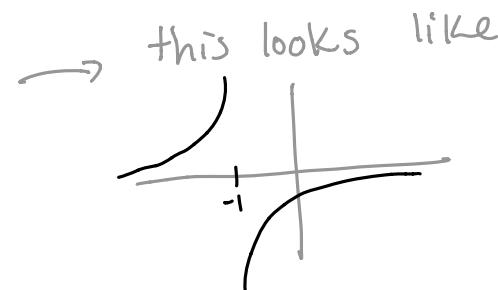
$$b) f(x) = \ln(1+x)$$

$$f'(x) = (1+x)^{-1} = \frac{1}{1+x}$$

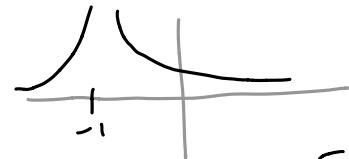
$$f''(x) = -(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3}$$

$$f''''(x) = -6(1+x)^{-4}$$



so  $|f^{(4)}|$  looks like



on the interval  $[1, 9]$

this is decreasing,

$$\text{so } |f^{(4)}(x)| \leq |f^{(4)}(1)| = \left| \frac{-6}{2^4} \right| = \frac{6}{2^4}$$

choose  
this as  $\approx$

$$|E_S| \leq \frac{\frac{6}{2^4}(9-1)^5}{180(4^4)}$$

17 False

18 True

19  $n \geq 8$ .

Note: you may think it's  $n \geq 7$ ,  
but to use Simpson's rule  $n$   
must be even. So  $n \geq 8$

20a

$$\begin{aligned} \int_0^\infty xe^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t xe^{-x} dx \\ u = x &\quad dv = e^{-x} dx \\ du = dx &\quad v = -e^{-x} \\ &= \lim_{t \rightarrow \infty} \left( -xe^{-x} \Big|_0^t - \int_0^t -e^{-x} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( -xe^{-x} \Big|_0^t - e^{-x} \Big|_0^t \right) \\ &= \lim_{t \rightarrow \infty} \left( (-te^{-t} - e^{-t}) - \left( \underbrace{-0e^0}_0 - \underbrace{-e^0}_1 \right) \right) \\ &= \lim_{t \rightarrow \infty} \left( \frac{-t}{e^t} - \frac{1}{e^t} + 1 \right) \\ &= \lim_{t \rightarrow \infty} \frac{-t}{e^t} - \lim_{t \rightarrow \infty} \frac{1}{e^t} + 1 \\ &= \lim_{t \rightarrow \infty} \frac{-t}{e^t} - \lim_{t \rightarrow \infty} \frac{1}{e^t} + 1 \\ &= 0 - 0 + 1 = \boxed{1} \end{aligned}$$

20b

$$\int_{-5}^1 \frac{1}{10+2x} dx$$

$$\begin{aligned} u &= 10+2x && \text{when } x=1 \\ du &= 2 dx && u = 10+2(1) = 12 \\ \frac{1}{2} du &= dx && \text{when } u=-5 \\ &&& u=0 \end{aligned}$$

$$\int_0^{12} \frac{1}{u} du$$

$$= \lim_{t \rightarrow 0^+} \int_t^{12} \frac{1}{u} du$$

$$= \lim_{t \rightarrow 0^+} \left[ \ln|u| \right]_t^{12}$$

$$= \lim_{t \rightarrow 0^+} \ln(12) - \ln(t)$$

$$= \ln(12) - (-\infty)$$

$$= \infty$$

this integral Diverges

(20c)

$$\int_2^\infty \frac{9}{(6-3x)^4} dx$$

$$\left. \begin{array}{l} u = 6-3x \\ du = -3dx \\ -\frac{1}{3}du = dx \end{array} \right| \begin{array}{l} \text{when } x=\infty \\ u=6-3(\infty)=-\infty \\ \text{when } x=2 \\ u=6-3(2)=0 \end{array}$$

$$= \int_0^{-\infty} -\frac{1}{3} \frac{9}{u^4} du$$

$$= \int_{-\infty}^0 \frac{3}{u^4} du$$

consider  $\int_{-\infty}^{-1} \frac{3}{u^4} du$

$$= \lim_{t \rightarrow -\infty} \int_t^{-1} \frac{3}{u^4} du$$

$$= \lim_{t \rightarrow -\infty} -u^{-3} \Big|_t^{-1}$$

$$= \lim_{t \rightarrow -\infty} -(-1)^{-3} + t^{-3}$$

$$= 1 + 0$$

AND  $\int_{-1}^0 \frac{3}{u^4} du$

$$= \lim_{t \rightarrow 0} \int_{-1}^t \frac{3}{u^4} du$$

$$= \lim_{t \rightarrow 0} -u^{-3} \Big|_{-1}^t$$

$$= \lim_{t \rightarrow 0} -t^{-3} + (-1)^{-3}$$

$$= -\infty + 1$$

$$= -\infty$$

Since one of these diverges, then

$$\int_2^\infty \frac{9}{(6-3x)^4} dx$$

Diverges

(20d)

$$\int_0^\infty \sin x \, dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t \sin x \, dx$$

$$= \lim_{t \rightarrow \infty} -\cos x \Big|_0^t$$

$$= \lim_{t \rightarrow \infty} -\cos(t) + \cos(0)$$

↑  
limit DNE

so  $\int_0^\infty \sin x \, dx$  Diverges

(21)

$$\int_0^8 \frac{1}{\sqrt[3]{x}} \, dx = \lim_{t \rightarrow 0^+} \int_t^8 x^{-1/3} \, dx$$

$$= \lim_{t \rightarrow 0^+} \left. \frac{3}{2} x^{2/3} \right|_t^8$$

$$= \lim_{t \rightarrow 0^+} \frac{3}{2} (8^{2/3}) - \frac{3}{2} t^{2/3}$$

$$= \frac{3}{2} (8^{2/3}) - 0$$

$$= \frac{3}{2} (4)$$

$$= \boxed{6}$$

(22)

This work applies the Fundamental Theorem of Calculus (FTC)

However, we can only use FTC with continuous functions.

$\frac{1}{(x-2)^2}$  is NOT continuous.

(23)

Note: If  $\int_{-1}^0 \frac{1}{x^3} dx$  AND  $\int_0^1 \frac{1}{x^3} dx$  converge

$$\begin{aligned} \text{then } \int_{-1}^1 \frac{1}{x^3} dx &= \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx \\ &= -\int_0^1 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx = 0 \end{aligned}$$

$$\begin{aligned} \text{However } \int_{-1}^0 \frac{1}{x^3} dx &= \lim_{t \rightarrow 0} \int_{-1}^t \frac{1}{x^3} dx \\ &= \lim_{t \rightarrow 0} -\frac{1}{2} x^{-2} \Big|_{-1}^t \\ &= \lim_{t \rightarrow 0} -\frac{1}{2} t^{-2} + \frac{1}{2} (-1)^{-2} \\ &= -\infty + \frac{1}{2} \\ &= -\infty \end{aligned}$$

so this Diverges

Thus,  $\int_{-1}^1 \frac{1}{x^3} dx$  Diverges

## You have found the secret Bonus Question!

If you have an answer, write your work neatly on a piece of paper, and attach an image of your work in an email to me ([epavlec@ncsu.edu](mailto:epavlec@ncsu.edu)) with the subject title: 'Secret Bonus YourLastname'.

(for example, mine would be: 'Secret Bonus Pavlechko')

Correct & justified answers will receive a bonus point on Test #2.