NC STATE Math 241-003 - Study Guide - Fall 2021

Chapter 4

4.1: Sequences

- Determine if a sequence is convergent or divergent.
  (Refresh yourself on the Squeeze Theorem, L'Hopital's Rule, and your Limit Laws)
- Rewrite a list of terms using sequence notation.

4.2: Intro to Series

- Rewrite a given series using summation notation
- Define the k-th partial sum.
- Know when a geometric series is convergent and what it converges to
- Know about Telescoping Series
- Know when a p-series converges and when it diverges
- Know about the Harmonic Series
- Know the Divergence Test

4.3: Series Tests

- Know when the Integral Test, Direct Comparison Test, Limit Comparison Test can be applied.
- Perform the Integral Test, Direct Comparison Test, and/or Limit Comparison Test.
- Draw conclusions based on the tests you've performed.
- Know the Integral Test Estimation Theorem

4.4: Alternating Series

- Recognize an Alternating Series.
- Be able to perform the Alternating Series Test.
- Know the Alternating Series Estimation Theorem

4.5: Absolute Convergence

- Recognize when a series converges, absolutely converges, conditionally converges, or diverges.
- Perform the Absolute Convergence Test
- Perform the Ratio Test

4.6: Power Series

- Know what a Power Series is.
- Be able to find the radius, interval of convergence, and center for a given Power Series

Note. If we <u>do not</u> tell you which test to use, 'Showing your work' consists of:

(1) Identify the test you'd like to attempt

(2) Check the conditions of the test are satisfied
(i.e. Check you can use the test)

(3) Perform the test

(4) Draw conclusions about convergence/divergence.
We should see: "Using the ______________ Test, the series ______________."

Note. If we tell which test to use, 'Showing your work' consists of:

(1) Perform the test

(2) Draw conclusions about convergence/divergence.
We should see: "Using the ______________ Test, the series ______________."

(In other words, you don't need to check the conditions of the test)

<u>Exercises</u>: (The level of difficulty of the *Challenge* questions will not be on your test)

1. For each of the following, determine whether the sequence $(a_n)_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

   (a) $a_n = \cos(\frac{1}{n^2+1})$

   (b) $a_n = \frac{\ln(n+2)}{\ln(1+4n)}$

   (c) $a_n = \frac{(-3)^n}{(20)^{n+1}}$

   (d) $a_n = (-1)^n \frac{1}{n(n+4)}$

   (e) $a_n = (-1)^{n+1} \frac{5n}{10n+2}$

   (f) $a_n = \frac{2n^2+5}{n^2+n}$

2. For each of the following, does the sequence below converge or diverge. If it converges, find the limit.

   (a) $(4, 16, 64, 256, ...)$

   (b) $\left(\frac{-1}{7}, \frac{-1}{14}, \frac{-1}{21}, \frac{-1}{28}, ...\right)$

   (c) $\left(-\frac{5}{2}, \frac{10}{5}, -\frac{15}{10}, \frac{20}{17}, ...\right)$

3. (Challenge) If the partial sum $s_k = \frac{k^2}{5+2k}$, determine if the series $\sum_{n=1}^{\infty} a_n$ converges or diverges.

4. (Challenge) If the partial sum $s_k = \frac{5+8k^2}{2-7k^2}$, determine if the series $\sum_{n=1}^{\infty} a_n$ converges or diverges.

5. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

   (a) Does the series converge or diverge using the Integral Test?

   (b) Find how many terms are needed to ensure the remainder is less than 10^{-3}.

6. Consider the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$
   (a) Show that this series converges using the Alternating Series Test
   (b) How many terms must be summed so that the remainder is less than 10^{-4}?

7. Consider the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+1}$$
   (a) Show that this series converges using the Alternating Series Test
   (b) How many terms must be summed so that the remainder is less than 10^{-3}?

8. Do the following series converge or diverge? If it converges, state where it converges to.

   (a) $\frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \cdots$

   (b) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$.
   (Hint: $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$)

   (c) $\frac{-1}{7} + \frac{-1}{14} + \frac{-1}{21} + \frac{-1}{28} + \cdots$

   (d) $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$

9. Decide which test for convergence is appropriate and then use it to determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{3ne^n}{n^2+1}$

   (b) $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$

   (c) $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{5^n}$

   (d) $\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2(n)}$

   (e) $\sum_{n=1}^{\infty} \frac{4n^2-n}{n^3+9}$

   (f) $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$

   (g) $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

   (h) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

   (i) $\sum_{n=1}^{\infty} \frac{1}{5^n+7}$

   (j) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

10. Determine whether the following converges absolutely, converges conditionally, or diverges.

   (a) $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

11. Consider the power series $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

    (a) What is the Interval of convergence?

    (b) What is the radius of convergence?

    (c) What is the center of the power series?

12. Consider the power series $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$

    (a) What is the Interval of convergence?

    (b) What is the radius of convergence?

    (c) What is the center of the power series?

13. Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

    (a) What is the Interval of convergence?

    (b) What is the radius of convergence?

    (c) What is the center of the power series?

14. Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{\sqrt{n}4^n}$

    (a) What is the Interval of convergence?

    (b) What is the radius of convergence?

    (c) What is the center of the power series?

SOLUTIONS

1. (a) Converges to 1
   (b) Using L'Hopitals Rule, this converges to 1
   (c) Converges to 0
   (d) Using the Squeeze Theorem, this converges to 0
   (e) Diverges
   (f) Using L'Hopitals Rule, this converges to 2.
2. (a) Diverges.
   (b) Converges to 0.
   (c) Using the Squeeze Theorem, this converges to 0.
3. It Diverges
4. It Converges to $-\frac{8}{7}$
5. (a) This series converges by the Integral Test.
   (b) $n = 4,000,001$ terms
6. (a) This series converges by the Alternating Series Test.
   (b) $n = 4999$ terms
7. (a) This series converges by the Alternating Series Test.
   (b) $n = 5$ terms
8. (a) This is a geometric series with $a = \frac{5}{3}$ and $r = \frac{1}{3}$. Thus, this converges to $\frac{5}{2}$.
   (b) This series is a telescoping series, but it diverges to $-\infty$.
   (c) This series is a constant multiple of the harmonic series, so it diverges.
   (d) This series is a telescoping series, which converges to 4.

9. Note: You may be able to use multiple tests on each question. I've written the most *common* solution to each, but if you have an alternative solution and want to check it send me an email.

   (a) Using the Divergence Test, this series diverges.

   (b) Using the Direct Comparison Test, this series diverges. (Comparison used: Geometric series with $r = \frac{4}{3}$)
   Using the Limit Comparison Test, this series diverges. (Comparison used: Geometric series with $r = \frac{4}{3}$)

   (c) By the Ratio Test, this series converges.
   Also, by the Alternating Series Test this series converges.

   (d) Using the Direct Comparison Test, this series diverges. (Commparison used: Harmonic Series)
   Note: the Limit Comparison Test would be *inconclusive* here.

   (e) Using the Limit Comparison Test, this series diverges. (Comparison used: Harmonic Series)
   Note: You should NOT use the Direct Comparison Test, since $\frac{4n^2-n}{n^3+9}$ may not be greater than $\frac{4}{n} = \frac{4n^2}{n^3}$.

   (f) Using the Limit Comparison Test, this series converges. (Comparison used: Geometric series with $r = \frac{1}{3}$)
   Note: You should NOT use the Direct Comparison Test, since $\frac{1}{3^n-2} \geq \frac{1}{3^n}$.

   (g) Using the Ratio Test, this series converges

   (h) Using the Direct Comparison Test, this series converges. (Comparison used: p-series with $p = 2$)
   Using the Limit Comparison Test, this series converges. (Comparison used: p-series with $p = 2$)

   (i) Using the Direct Comparison Test, this series converges. (Comparison used: Geometric series with $r = \frac{1}{5}$)
   Using the Limit Comparison Test, this series converges. (Comparison used: Geometric series with $r = \frac{1}{5}$)

   (j) By the Integral Test, this series diverges.

10. (a) Using the Ratio Test, this series converges absolutely.

   (b) Using the Direct Comparison, Limit Comparison, or Integral Test this series does NOT converge absolutely. However, using the Alternating Series Test this series converges conditionally.

11. (a) $(-2, 2)$
    (b) $R = 2$
    (c) center $=$ 0
12. (a) $[0, 0]$
    (b) $R = 0$
    (c) center $=$ 0
13. (a) $(0, 2]$
    (b) $R = 1$
    (c) center $=$ 1
14. (a) $(-6, 2]$
    (b) $R = 4$
    (c) center $=$ -2