

Directions: Write your full name and Test Form (A or B) on the front of the Blue Book. In the *Box No.* fill in your *Row Letter, Seat Number*.

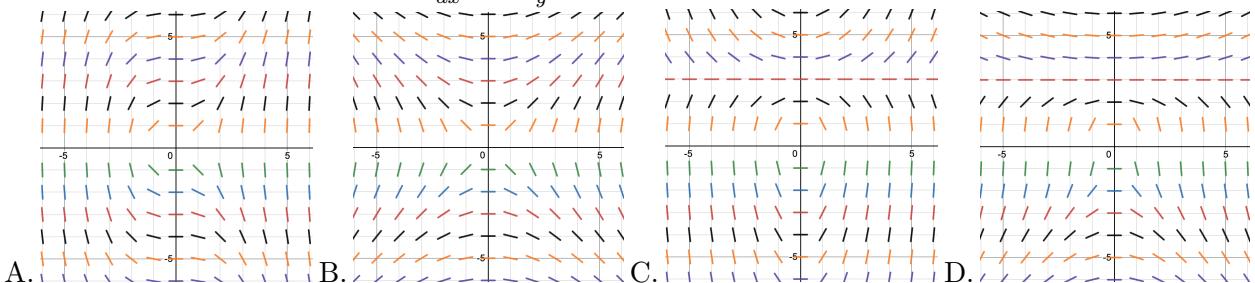
All work must be shown in the Blue Book to receive credit, and only work in the Blue Book will be graded.

Number all questions, including parts, and box your final answers.

No phones, notes, calculators, or other aids are allowed.

1. Find the equilibrium solutions of $\frac{dy}{dx} = \frac{x(y-3)}{y}$ and match it to its slope field.

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2. Use the Initial Value Problem: $\frac{dy}{dx} = \frac{8x+12}{e^y}$, $y(1) = 0$ to answer the following:

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- (a) Use Euler's method with a stepsize of 0.1 to approximate $y(1.2)$.
- (b) Solve the IVP. Give your answer with y as an explicit function of x if possible.

3. Find the orthogonal trajectories of $y = kx^2$.

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4. A population of rabbits grows at a rate proportional to its size. Suppose initially, we have 6 rabbits and after 30 days we have 24 rabbits. Find an equation for the number of rabbits after t days.

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5. The air in a tank with volume 210 m^3 contains 2% laughing gas by volume initially. Fresher air with only 1% laughing gas flows into the tank at a rate of $3 \text{ m}^3/\text{min}$ and the mixed air flows out at the same rate.

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- (a) Find the amount of laughing gas in the tank as a function of time.
- (b) A mixture of 5% laughing gas begins to affect humans. How long (in minutes) will it take for the laughing gas in the tank to reach this level?

6. Consider $y'' + 5y' = F(x)$.

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- (a) Find the general solution if $F(x) = 0$
- (b) Find the general solution if $F(x) = 8$.

7. Find the solution to $y'' + 2y' + 10y = 100e^{3x}$ with $y(0) = 6$, $y'(0) = 13$.

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#1

$$\frac{dy}{dx} = \frac{x(y-3)}{y}$$

Equilibrium solutions: when $y=a$ constant

$$\frac{x(y-3)}{y} = 0 \Rightarrow x=0 \text{ or } \boxed{y=3}$$

(so it's graph C or D)

Test points:

$$\text{at } (1, -1) \text{ we have } \frac{dy}{dx} = 1 \frac{(-1-3)}{-1} = \frac{-4}{-1} = \textcircled{+}$$

C: at $(1, -1)$ is $\textcircled{+}$ ← so it's C

D: at $(1, -1)$ is $\textcircled{-}$

#2) a) $\frac{dy}{dx} = \frac{8x+12}{e^y}$

| x | y | $\frac{dy}{dx}$ |
|-----|-----------------------------|---|
| 1 | 0 | $\frac{8(1)+12}{e^0} = \frac{20}{1} = 20$ |
| 1.1 | $0 + 20(0.1) = 2$ | $\frac{8(1.1)+12}{e^2} = \frac{8.8+12}{e^2} = \frac{20.8}{e^2}$ |
| 1.2 | $2 + \frac{20.8}{e^2}(0.1)$ | |

so

$$y(1.2) \approx 2 + \frac{20.8}{e^2}(0.1)$$

Note: you can leave it like this on the test

b) $\frac{dy}{dx} = \frac{8x+12}{e^y}$

$$e^y dy = 8x+12 dx$$

$$\int e^y dy = \int 8x+12 dx$$

$$e^y = 8 \frac{x^2}{2} + 12x + C$$

$$y = \ln(4x^2 + 12x + C)$$

using $y(1)=0$:

$$0 = \ln(4(1)^2 + 12(1) + C)$$

$$0 = \ln(16 + C)$$

so $C = -15$

$$y = \ln(4x^2 + 12x - 15)$$

$$\textcircled{3} \quad y = kx^2 \quad \rightsquigarrow \quad k = \frac{y}{x^2}$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[kx^2]$$

$$\frac{dy}{dx} = k \cdot 2x$$

$$= \frac{y}{x^2} \cdot 2x = \frac{2y}{x}$$



$$\frac{dy}{dx} = -\frac{x}{2y}$$

$$y \, dy = -\frac{1}{2}x \, dx$$

$$\int y \, dy = \int -\frac{1}{2}x \, dx$$

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{4} + C \quad \text{or} \quad y^2 = -\frac{x^2}{2} + C}$$

$$\textcircled{#4} \quad y(t) = y_0 e^{kt}$$

$$y(0) = 6$$

$$y(30) = 24$$

Note: the phrasing
"grows at a rate proportional to
its size" indicates
exponential growth, not logistic

$$y(t) = 6e^{kt}$$

Find k :

$$24 = 6e^{k(30)}$$

$$4 = e^{30k}$$

$$\ln(4) = 30k$$

$$\frac{\ln(4)}{30} = k$$

$$y(t) = 6e^{\frac{\ln(4)}{30}t}$$

#5 $y(t)$ = amount of laughing gas in the tank at time t
(measured in m^3)

$t = \# \text{ mins.}$

$$y(0) = (\text{concentration})(\text{volume}) = \underbrace{(6.02)}_{2\%}(210) = 4.2 \text{ m}^3$$

$$\begin{aligned}\frac{dy}{dt} &= f_i c_i - f_o c_o \\ &= (3)(\underbrace{6.01}_{1\%}) - 3\left(\frac{y}{210}\right) \\ &= 6.03 - \frac{3y}{210} \\ \frac{dy}{dt} &= \frac{6.03 - 3y}{210}\end{aligned}$$

$$\frac{1}{6.03 - 3y} dy = \frac{1}{210} dt$$

$$\int \frac{1}{6.03 - 3y} dy = \int \frac{1}{210} dt$$

$$-\frac{1}{3} \ln |6.03 - 3y| = \frac{1}{210} t + C$$

$$\ln |6.03 - 3y| = \frac{-3}{210} t + \tilde{C} = -\frac{1}{70} t + \tilde{C}$$

$$|6.03 - 3y| = e^{-\frac{1}{70}t} e^{\tilde{C}}$$

$$6.03 - 3y = A e^{-\frac{1}{70}t}$$

$$-3y = A e^{-\frac{1}{70}t} - 6.03$$

$$y = \frac{A}{3} e^{-\frac{1}{70}t} + 2.1$$

using $y(0) = 4.2$:

$$4.2 = \frac{A}{-3} e^{-\frac{1}{70}(0)} + 2.1$$

$$2.1 = \frac{A}{-3}$$

$$-6.3 = A$$

$$y(t) = \frac{-6.3}{-3} e^{-\frac{1}{70}t} + 2.1$$

a) $\boxed{y(t) = 2.1 e^{-\frac{1}{70}t} + 2.1}$

b) amount = $\underbrace{(6.05)}_{5\%} (210) = 10.5$

$$10.5 = 2.1 e^{-\frac{1}{70}t} + 2.1$$

$$8.4 = 2.1 e^{-\frac{1}{70}t}$$

$$4 = e^{-\frac{1}{70}t}$$

$$\ln(4) = -\frac{1}{70}t$$

$\boxed{-70 \ln(4) = t} \approx 97.04 \text{ mins.}$

#6

$$a) y'' + 5y' = 0$$

$$r^2 + 5r = 0$$

$$r(r+5) = 0$$

$$r=0 \text{ or } r=-5$$

$$y = C_1 e^{0x} + C_2 e^{-5x} = \boxed{C_1 + C_2 e^{-5x}}$$

$$b) y'' + 5y' = 8$$

$$y_p = C_1 + C_2 e^{-5x}$$

$$y_p = Ax \quad (\text{Note: It can't be just } A, \text{ since that would "overlap" } y_c)$$

$$y_p' = A \quad \text{so} \quad y_p'' + 5y_p' = 8$$

$$0 + 5A = 8$$

$$\boxed{y = C_1 + C_2 e^{-5x} + \frac{8}{5}x} \quad A = \frac{8}{5}$$

$$\textcircled{#7} \quad y'' + 2y' + 10y = 100e^{3x}$$

$$r^2 + 2r + 10 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$$y_c = e^{-x} [C_1 \cos(3x) + C_2 \sin(3x)]$$

$$y_p = Ae^{3x} \quad (\text{No overlap with } y_c \checkmark)$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y_p'' + 2y_p' + 10y_p = 100e^{3x}$$

$$9Ae^{3x} + 2(3Ae^{3x}) + 10(Ae^{3x}) = 100e^{3x}$$

$$(9A + 6A + 10A)e^{3x} = 100e^{3x}$$

$$25Ae^{3x} = 100e^{3x}$$

$$A = 4$$

$$y_p = 4e^{3x}$$

$$y = e^{-x} [C_1 \cos(3x) + C_2 \sin(3x)] + 4e^{3x}$$

$$\text{use } y(0) = 6$$

$$6 = e^0 [C_1 \cos(0) + \cancel{C_2 \sin(0)}] + 4e^0$$

$$6 = C_1 + 4$$

$$2 = C_1$$

$$\text{use } y'(0) = 13$$

$$y' = -e^{-x} [2\cos(3x) + C_2 \sin(3x)] + e^{-x} [-6\sin(3x) + 3C_2 \cos(3x)] + 12e^{3x}$$

$$\begin{aligned} 13 &= -e^0 [2\cos(0) + C_2 \sin(0)] \\ &\quad + e^0 [-6\sin(0) + 3C_2 \cos(0)] + 12e^0 \end{aligned}$$

$$13 = -2 + 3C_2 + 12$$

$$3 = 3C_2$$

$$1 = C_2$$

$$y = e^{-x} [2\cos(3x) + \sin(3x)] + 4e^{3t}$$