MA 241: Exam 4

Practice exam

This exam is worth a total of **100 points**.

Directions: Show all work in a clear/logical manner and justify all conclusions in order to receive full credit. Number all questions, including parts, and box your final answers.
No calculators, collaboration with other students, notes, or internet usage is allowed.

1. Does the sequence $\left(\dfrac{n!}{(n+2)!}\right)_{n=1}^{\infty}$ converge? If so, what does it converge to? $\boxed{7}$

2. Do the following series converge or diverge? If it converges, state where it converges to. $\boxed{16}$

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5^{n-1}}{3^{n-2}}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n+3}-\frac{3}{n+4}\right)$

3. Decide which test for convergence is appropriate and then use it to determine whether the series is convergent or divergent. $\boxed{16}$

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{4n^2-n}{n^3+9}$

   (b) $\left\{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots\right\}$

4. (a) Determine if $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n+3^n}$ converges or diverges using the Alternating Series Test. $\boxed{15}$

   (b) Approximate the series using the first 4 terms

   (c) Estimate the error

5. (a) Determine if $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n+7)^3}$ converges or diverges using the Integral Test. $\boxed{15}$

   (b) Determine the least value of k such that R_k is less than $\frac{1}{784}$.

6. Determine if the following series converge absolutely, converge conditionally, or diverge: $\boxed{16}$

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{8^n}{(n+1)5^{2n-1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+4}}$

7. Consider the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{6^n(n^3+1)}$ $\boxed{15}$

   (a) What is the interval of convergence?

   (b) What is the radius of convergence?

   (c) What is the center of the power series?

#1 Note: $\left(\frac{n!}{(n+2)!}\right)_{n=1}^{\infty}$ is a <u>sequence</u> NOT a series!

$$\lim_{n\to\infty} \frac{n!}{(n+2)!} = \lim_{n\to\infty} \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(n+2)(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= \lim_{n\to\infty} \frac{1}{(n+2)(n+1)}$$

$$= 0$$

So the <u>sequence converges to 0.</u>

#2 a) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5^{n-1}}{3^{n-2}} = \sum_{n=1}^{\infty} (-1)^{n-1}(-1)^2 \frac{5^{n-1}}{3^{n-1}\,3^{-1}}$$

$$= \sum_{n=1}^{\infty} (-1)^2 \frac{1}{3^{-1}} \left(-\frac{5}{3}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 3 \left(-\frac{5}{3}\right)^{n-1}$$

Geometric Series, $a=3$, $r=-\frac{5}{3}$

Since $|r| = \left|-\frac{5}{3}\right| > 1$ this series

$\boxed{\text{diverges}}$

b) $$\sum_{n=1}^{\infty} \left(\frac{3}{n+3} - \frac{3}{n+4}\right)$$ telescoping?

$$S_k = \left(\frac{3}{4} - \cancel{\frac{3}{5}}\right) + \left(\cancel{\frac{3}{5}} - \cancel{\frac{3}{6}}\right) + \dots + \left(\cancel{\frac{3}{k+2}} - \cancel{\frac{3}{k+3}}\right) + \left(\cancel{\frac{3}{k+3}} - \frac{3}{k+4}\right)$$

$$= \frac{3}{4} - \frac{3}{k+4}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{3}{4} - \frac{3}{k+4} = \frac{3}{4}$$

The series converges to $\boxed{\frac{3}{4}}$.

#3 a) $\sum_{n=1}^{\infty} \frac{4n^2-n}{n^3+9}$

Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{4n^2}{n^3} = \sum_{n=1}^{\infty} \frac{4}{n}$ ← Harmonic diverges

$$\frac{4n^2-n}{n^3+9} = \frac{\overset{+}{n}\,\overbrace{(4n-1)}^{+}}{\underbrace{n^3+9}_{+}}$$

so it's positive for $n \geq 1$

$$\lim_{n\to\infty} \frac{\left(\frac{4n^2-n}{n^3+9}\right)}{\left(\frac{4}{n}\right)} = \lim_{n\to\infty} \frac{(4n^2-n)n}{4(n^3+9)} = 1$$

since $1 < \infty$, by the Limit Comparison Test this series $\boxed{\text{diverges}}$

Note: You wouldn't be able to draw conclusions from Direct Comparison Test on this series.

b) $\left\{ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots \right\} = \sum_{n=1}^{\infty} \frac{n}{n+1}$

Divergence Test

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

This series $\boxed{\text{diverges}}$ using the Divergence Test

#4 a) $c_n = \frac{1}{2^n+3^n}$

$$\frac{1}{2^1+3^1} > \frac{1}{2^2+3^2} + \frac{1}{2^3+3^3} + \dots$$

so c_n is decreasing ✓

$$\lim_{n\to\infty} \frac{1}{2^n+3^n} = 0$$ ✓

By the Alternating Series Test, this converges

b) $$S_4 = \sum_{n=1}^{4} (-1)^n \frac{1}{2^n+3^n} = \underbrace{\frac{1}{2^1+3^1} + \frac{1}{2^2+3^2} + \frac{1}{2^3+3^3} + \frac{1}{2^4+3^4}}$$

Don't need to plug this into calculator

c) $|R_N| \leq c_{N+1}$

$$|R_4| \leq c_5 = \boxed{\frac{1}{2^5+3^5}}$$

#5 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+7)^3}$

a) $\displaystyle\int_1^{\infty} \frac{1}{(2x+7)^3}\,dx$

$u = 2x+7$ | when $x = \infty$
$du = 2\,dx$ | $u = 2(\infty)+7 = \infty$
$\frac{1}{2}\,du = dx$ | when $x = 1$
| $u = 2(1)+7 = 9$

$$= \int_9^{\infty} \frac{1}{u^3}\,\frac{1}{2}\,du$$

$$= \frac{1}{2} \lim_{t\to\infty} \int_9^{t} u^{-3}\,du$$

$$= \frac{1}{2} \lim_{t\to\infty} \left[\frac{u^{-2}}{-2}\right]_9^{t}$$

$$= -\frac{1}{4}\left(\lim_{t\to\infty} t^{-2} - 9^{-2}\right)$$

$$= -\frac{1}{4}\left(0 - \frac{1}{81}\right) = \frac{1}{4}\left(\frac{1}{81}\right) = \frac{1}{324}$$

since $\displaystyle\int_1^{\infty} \frac{1}{(2x+7)^3}\,dx$ converges, by the integral test,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+7)^3}$ converges.

b) $\int_{k+1}^{\infty} \frac{1}{(2x+7)^3}\,dx \le R_k \le \int_{k}^{\infty} \frac{1}{(2x+7)^3}\,dx < \frac{1}{784}$

using the work from part a)

$u = 2x+7$

$$\frac{1}{2} \lim_{t\to\infty} \left[\frac{u^{-2}}{-2}\right]_{2k+7}^{t} < \frac{1}{784}$$

$$-\frac{1}{4}\left(\lim_{t\to\infty} t^{-2} - (2k+7)^{-2}\right) < \frac{1}{784}$$

$$\frac{1}{4}\left(\frac{1}{(2k+7)^2}\right) < \frac{1}{784}$$

$$\frac{1}{(2k+7)^2} < \frac{4}{784} = \frac{1}{196}$$

$$(2k+7)^2 > 196$$

$$2k+7 > \underbrace{\sqrt{196}}_{14}$$

$$2k > 14-7$$

$$k > 3.5$$

so $\boxed{k=4}$ makes $R_k < \frac{1}{784}$.

#6 a) $\sum_{n=1}^{\infty} \frac{8^n}{(n+1)5^{2n-1}}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{8^{n+1}}{(n+2)5^{2(n+1)-1}}}{\frac{8^n}{(n+1)5^{2n-1}}} \right| = \lim_{n\to\infty} \left| \frac{8^{n+1} \; 5^{2n-1} (n+1)}{(n+2)5^{2n+1} \; 8^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{8 \; (n+1)}{(n+2) \; 5^2} \right|$$

$$= \frac{8}{5^2} = \frac{8}{25}$$

since $\frac{8}{25} < 1$ this series <u>converges absolutely</u> by the Ratio test.

6 b) Check $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2+4}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ ← Harmonic series diverges.

$\frac{1}{\underbrace{\sqrt{n^2+4}}_{+}}$ is positive for $n \geq 1$ ✓

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+4}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = 1$$

since $1 \neq \pm\infty$ the $\sum \left|\frac{(-1)^n}{\sqrt{n^2+4}}\right|$ diverges using Limit Comparison Test

Alternating Series Test with $C_n = \frac{1}{\sqrt{n^2+4}}$

$$\frac{1}{\sqrt{1+4}} > \frac{1}{\sqrt{4+4}} > \frac{1}{\sqrt{9+4}} > \dots$$

so decreasing ✓

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+4}} = 0 \quad ✓$$

so the series converges by Alternating Series Test.

Thus, the $\boxed{\text{series is conditionally convergent}}$

#7

$$\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{6^{n+1}((n+1)^3+1)}}{\frac{x^n}{6^n(n^3+1)}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}\,6^n(n^3+1)}{6^{n+1}((n+1)^3+1)\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-1)(n^3+1)}{6((n+1)^3+1)}\right|$$

$$= \left|\frac{x-1}{6}\right|$$

a) $\left|\frac{x-1}{6}\right| < 1$

$|x-1| < 6$

$-6 < x-1 < 6$

$-5 < x < 7$

when $x=7$: $\sum_{n=1}^{\infty} \frac{6^n}{6^n(n^3+1)} = \sum \frac{1}{n^3+1}$

Direct comparison with $\sum \frac{1}{n^3}$ convergent p-series

$\frac{1}{n^3+1}$ is positive

since $n^3+1 \geq n^3$

$\frac{1}{n^3+1} \leq \frac{1}{n^3}$ ✓

so this converges by Direct Comparison Test when $x = 5$: $\sum_{n=1}^{\infty} \frac{(-6)^n}{6^n(n^3+1)} = \sum \frac{(-1)^n}{n^3+1}$

Absolute Convergence Test

since $\sum \left|\frac{(-1)^n}{n^3+1}\right| = \sum \frac{1}{n^3+1}$ converges

then, $\sum \frac{(-1)^n}{n^3+1}$ converges by absolute convergence

$\boxed{\text{Interval of Convergence: } [-5, 7]}$

b) $\boxed{R = 6}$

c) $\boxed{\text{center: } 1}$